

# Fourier Transform Optics

Bin LI

Dept. of Physics & Astronomy, Univ. of Pittsburgh, Pittsburgh, PA 15260, U. S. A,

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## Introduction

First, let me introduce some utilities in the following diagram.

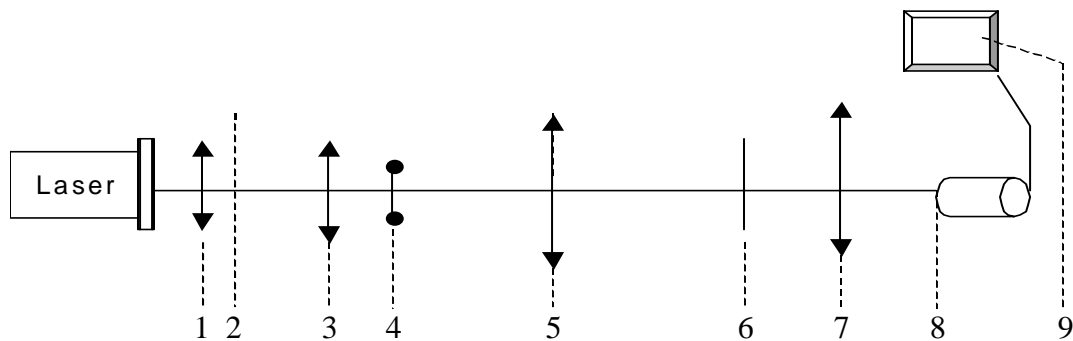


Fig. 1

- 1&3: The first converge lens ----- Lens 1, working along with Lens 3 as an light expander, expands the thin Laser Source to a much wider parallel light source.
- 2: The image focal plane of Lens 1; coincidentally, it is also the object focal plane of Lens 3.
- 4: The object, which is located in the object focal plane of the Lens 5 (Here Lens 5 is essential in this experiment, it works as the 1<sup>st</sup> Fourier Transform Equipment.)
- 5: Lens 5, the 1<sup>st</sup> Fourier Transform Lens.

- 7: Lens 7, the 2<sup>nd</sup> Fourier Transform Lens.
- 6: Filter Domain, which is not only the image focal plane of the 1<sup>st</sup> Fourier Transform Lens (Lens 5), but also the object focal plane of the 2<sup>nd</sup> Fourier Transform Lens (Lens 7). It is also called the 1st Fourier Transform Plane, since we can consider that object (4) in the focal plane of Lens 5 is Fourier Transformed into the other focal plane of Lens 5.
- 8: Fast CCD camera, which is used to take pictures in the image focal plane of the 2<sup>nd</sup> Fourier Transform Lens (Lens 7). This focal plane is also called the 2<sup>nd</sup> Fourier Transform Plane, since the light from Object 4 can arrive here after twice Fourier Transformations.
- 9: Computer, using the soft ware ----- *Quick Cam* to control of picture taken.

For simplicity, Let's discuss the problem in the view of "Geometrical Optics" first. Referring to Figure. 1, we know the transmitted light from Lens 3 will be a gird of parallel light with a finite radius -----  $r$ , it will shine Object 4 homogenously. So each point of Object 4 can be looked as a point source. Then, the light will start from the object as spherical wave, and cross the other two lenses in the set-up sequence. From the following picture (Figure. 2), which is just partial configures' "zoom in" of Figure. 1, we can see after light crosses Lens 5 and Lens 7, the image of object, which is real and inverted, will appear in the 2<sup>nd</sup> Fourier Transform Plane. If we put our CCD camera there, we can take a picture for the image.

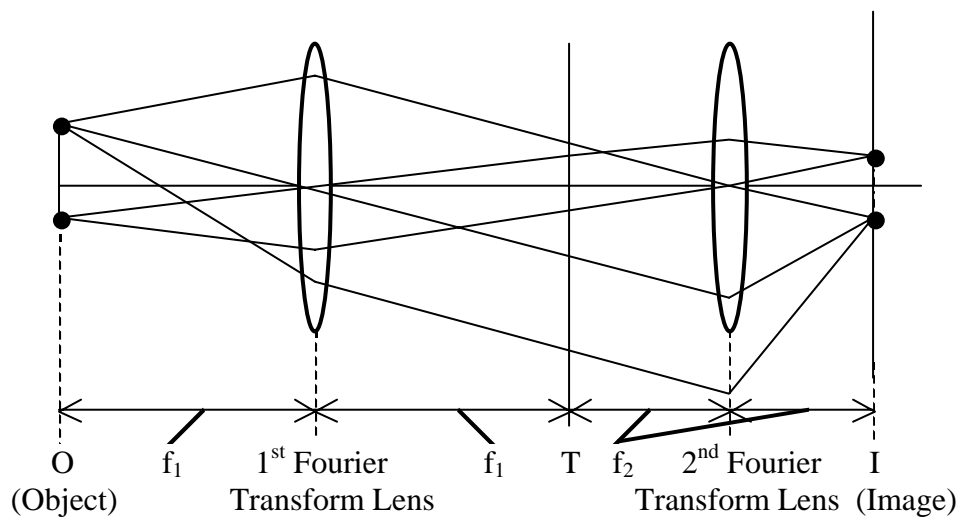


Fig. 2

Now, let's discuss this problem more seriously, by using the theory of "Fourier Transform Optics". Generally, in optical system, Lens has two major functions:

- (1) It works as an aperture, it can constraint both the incident wave front and the refraction wave front since it has a finite surface area.
- (2) It can also change the wave front. After a wave front crossing a lens, its amplitude and phase factor will change.

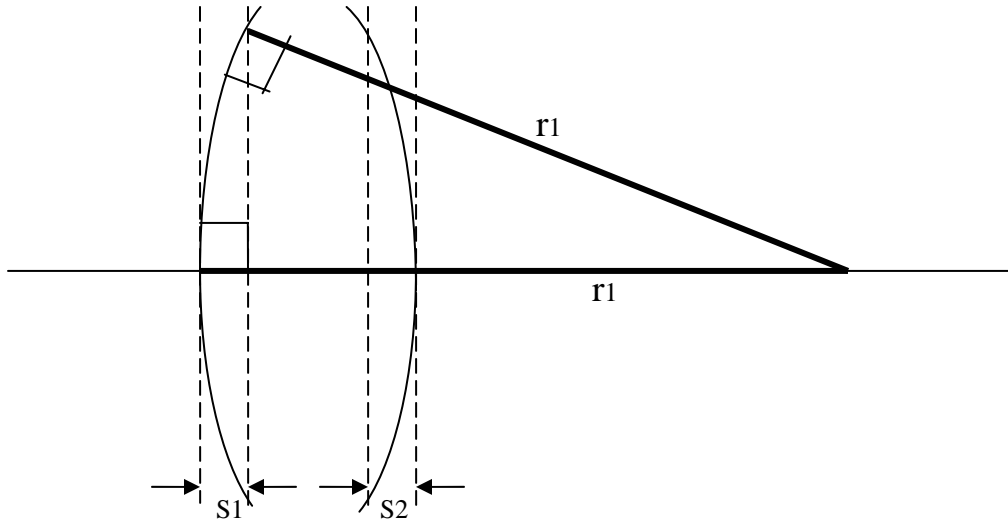
Assuming that we have an incident wave with a complex wave function just before the lens:

$$\tilde{U}_1(x, y) = A_1(x, y) \cdot \exp[i\varphi_1(x, y)].$$

Right after it crosses the lens, the wave function for the refraction wave becomes:

$$\tilde{U}_2(x, y) = A_2(x, y) \cdot \exp[i\varphi_2(x, y)]$$

(Please refer to Figure. 3.)



**Fig. 3**

So the aperture function of the lens is:

$$\tilde{t}_L(x, y) = \frac{\tilde{U}_2(x, y)}{\tilde{U}_1(x, y)} = \frac{A_2}{A_1} \cdot \exp[i(\varphi_2 - \varphi_1)] = \begin{cases} a(x, y) \cdot e^{i\varphi_L(x, y)}, & (r < \frac{D}{2}) \\ 0, & (r > \frac{D}{2}) \end{cases}$$

Where D represents the diameter of the lens,  $r = \sqrt{x^2 + y^2}$  is the distance from the incident point to the optic axis of the lens. For ideal case, the absorption of the lens is

negligible,  $a(x, y) = \frac{A_2}{A_1} \approx 1$ , so the aperture function becomes pure phase function, when

$r < \frac{D}{2}$ ,  $\tilde{t}_L(x, y) = e^{i\varphi_L(x, y)}$ . Strict calculation of phase function  $\varphi_L(x, y)$  is very difficult, so we have to use far field approximation for the object. We assume object is far from the lens, so the light emitting from object is approximately parallel to the lens's axis. Now, we can get that  $\varphi_L(x, y) = \frac{2\pi}{\lambda}[s_1 + s_2 + n \cdot d(x, y)] = -\frac{2\pi}{\lambda}(n-1) \cdot (s_1 + s_2) + \varphi_0$ , where  $\varphi_0 = \frac{2\pi}{\lambda}nd_0$ ,  $n$  is the index of the glass lens.

In general, the lens is thin lens, and the object is approaching the lens's axis.

$$\text{So we have: } s_1 = r_1 - \sqrt{r_1^2 - (x^2 + y^2)} \approx \frac{x^2 + y^2}{2r_1}$$

$$\text{And } s_2 = (-r_2) - \sqrt{(-r_2)^2 - (x^2 + y^2)} \approx -\frac{x^2 + y^2}{2r_2}$$

$r_1$  and  $r_2$  are the radius of the object surface and the image surface of the converge lens respectively. Because  $\varphi_0$  is a constant, so we can omit it in the expression of  $\varphi_L(x, y)$ .

Now, we get:

$$\varphi_L(x, y) = -\frac{2\pi}{\lambda} \cdot \frac{n-1}{2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) (x^2 + y^2) = -k \cdot \frac{x^2 + y^2}{2f}$$

In this equation,  $k$  is the wave number of the Laser light wave, and  $f$  is the focal length of the lens, since in geometrical optics, we know  $f = \frac{1}{(n-1) \cdot \left( \frac{1}{r_1} - \frac{1}{r_2} \right)}$ .

Now, we get the aperture transmission function of one lens:

$$\tilde{t}_L(x, y) = e^{i\varphi_L(x, y)} = \exp\left(-ik \frac{x^2 + y^2}{2f}\right).$$

Then, let's discuss about how the lens transforms the object in the Object Plane to the T-Plane. (Please refer to Figure. 2).

From Figure.2, we can see that the spherical wave from a point source of the object at coordinate  $(x_I, y_I, z_I)$  referent to the center of lens has a phase factor at an arbitrary place in space  $(x, y, z)$ :  $\exp\left[ik\left(\frac{x^2 + y^2}{2z_I} - \frac{xx_I + yy_I}{z_I}\right)\right]$ . Here, we still use the

approaching axis approximation and far field approximation as well, so the light can be assumed parallel to the optical axis.

So this special case,  $z = f$ , so we have incident wave function:

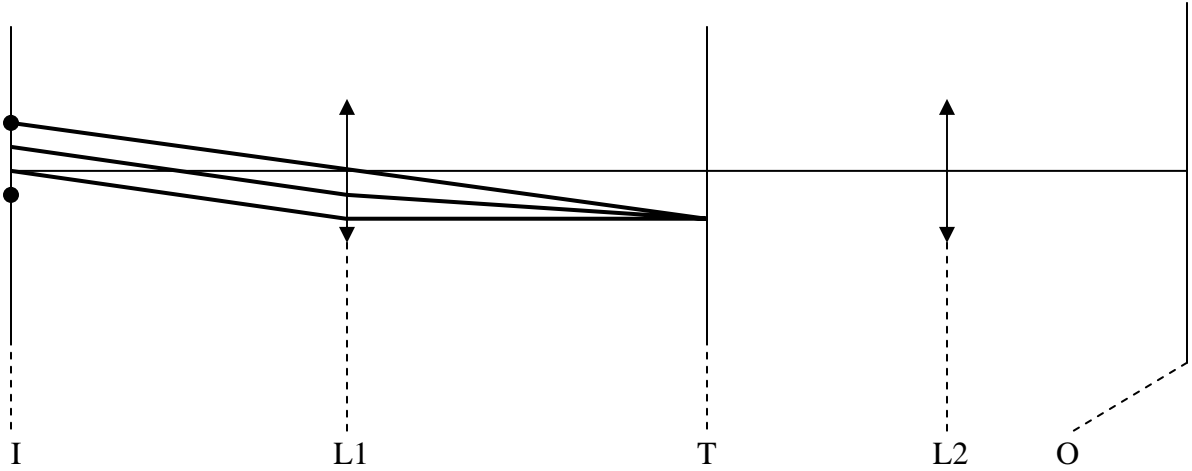
$$\tilde{U}_I(x, y, z) = U_I(x_I, y_I) \cdot \exp[ik(\frac{x^2 + y^2}{2f} - \frac{xx_I + yy_I}{f})]$$

After, the transformation of the first lens, the wave function in the T-Plane is:

$$\begin{aligned} \tilde{U}_T(x, y, z) &= \tilde{U}_I \cdot \tilde{t}_L = U_I(x_I, y_I) \cdot \exp[ik(\frac{x^2 + y^2}{2f} - \frac{xx_I + yy_I}{f})] \times \exp(-ik\frac{x^2 + y^2}{2f}) \\ &= U_I(x_I, y_I) \exp[-ik\frac{xx_I + yy_I}{f}]. \end{aligned}$$

But that's not the whole story, since the Laser Source is a coherent source, the other parallel components will also contribute to the transform configure in the T-Plane. (Please refer to Figure. 4) So actually, the wave function in the T-Plane will be the integral of all the contributions, and this effect really seems as a two dimensional Fourier Transformation:

$$\tilde{U}_T(x, y, z) = \int_{-\infty}^{+\infty} dx_I \int_{-\infty}^{+\infty} dy_I U_I(x_I, y_I) \exp[-ik\frac{xx_I + yy_I}{f}] \quad (1)$$



**Fig. 4**

Then the wave function from T-Plane will cross the 2<sup>nd</sup> Lens ----- L2, so there will be another Fourier Transformation, which will create an image in the O-Plane at  $(x_0, y_0, z_0)$ .

$$\begin{aligned}
\tilde{U}_o(x_0, y_0) &= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy U_T(x, y, z) \exp[-ik \frac{x_0 x + y_0 y}{f'}] \quad (2) \\
&= \int_{-\infty}^{+\infty} dx_I \int_{-\infty}^{+\infty} dy_I \{ \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy U_I(x_I, y_I) \exp[-ik \frac{(x_I + x_0 \frac{f}{f'})x + (y_I + y_0 \frac{f}{f'})y}{f}] \} \\
\therefore \tilde{U}_o(x_0, y_0) &= \int_{-\infty}^{+\infty} dx_I \int_{-\infty}^{+\infty} dy_I U_I(x_I, y_I) \delta(x_I + x_0 \frac{f}{f'}) \delta(y_I + y_0 \frac{f}{f'}) \\
&= U_I(-\frac{f}{f'} x_0, -\frac{f}{f'} y_0).
\end{aligned}$$

So we can see that, after twice Fourier Transformations, the image will be inverted, and it can be either magnified (if  $f > f'$ ), or reduced (if  $f < f'$ ).

In equation (1), if we let  $k_x = k \frac{x}{f}$  and  $k_y = k \frac{y}{f}$ , we will get:

$$\tilde{U}_T(x, y) = \int_{-\infty}^{+\infty} dx_I \int_{-\infty}^{+\infty} dy_I U_I(x_I, y_I) \exp[-ik_x x_I - ik_y y_I].$$

So in T-Plane, we can write the Fourier Transform configure as:

$$\tilde{U}_T(x, y) = \tilde{F}(k_x, k_y) = F\{\tilde{U}_I(x_I, y_I)\}.$$

In same way, in equation (2), we can let  $k_x' = -k \frac{x}{f'}$  and  $k_y' = -k \frac{y}{f'}$ , so the function of the 2<sup>nd</sup> lens can be looked as an Anti-Fourier Transformation:

$$\begin{aligned}
\tilde{U}_o(x_0, y_0) &= \int_{-\infty}^{+\infty} dk_x' \int_{-\infty}^{+\infty} dk_y' \cdot U_T(x, y) \exp[ik_x' x_0 + ik_y' y_0] \\
\tilde{U}_o(x_0, y_0) &= F^{-1}\{\tilde{U}_T(x, y)\}.
\end{aligned}$$

## Experimental Set-Up & Procedure

Our experiment set-up is exactly same as Figure.1, which we demonstrate at the beginning of this report.

Here we put two parallel linear polarizers in front of the Laser Source. We can change the relative orientations of these two polarizers to control the intensity of the

Laser Beam. Lens 1 and Lens 3 work corporately as a Light Expander. Lens 1 is a very small Lens and its focal length is also very small; Lens 3 has a focal Length of 200 mm. Lens 5 and Lens 7 are very important utilities in our experiment, both of them work as Fourier Transformation Lens. Their focal lengths are 200 mm and 100 mm respectively.

First, we should adjust the heights of all the utilities on the optical bench, try to make the Laser Beam goes through the centers of all the lenses. After Laser Beam expanding, the intensity should be homogenous as much as possible, and shines the object fully and uniformly. After that, we can fix the positions of equipment 1, 2, 3, 4.

Next, we adjust the positions and orientations of Lens 5 and Lens 7, till all the experimental conditions are satisfied. (Their focal planes overlap, and the expanding Laser gird crosses their centers.) Then we adjust the position of CCD camera till we get a clear image. After we get sharpening images, we can use soft ware ----- Quick Cam to control CCD camera to take pictures, and save them as .bmp files.

Basically, for each object, we should take three photo-pictures, two of them should be taken in the I-Plane (Image Plane), and the other one should be taken in the T-Plane (Transformation Plane, please refer to Fig. 2).

For those two images taken in the I-Plane: One, we should do nothing more; for the other one, we should use a spatial filter to get trid of some components of the patterns in the Fourier Transformation Plane, and see what change will appear in its corresponding image comparing with the image taken without filter.

## Results & Discussion

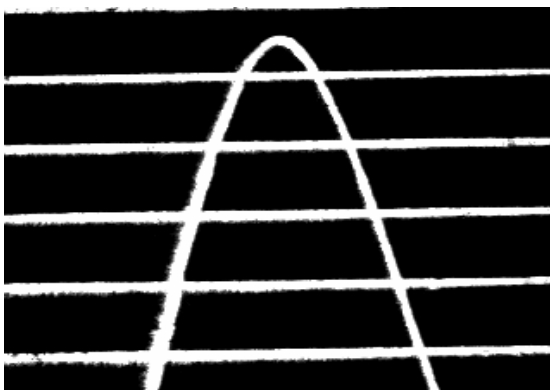


Fig. 5 ----- Image of Sine Wave with Parallel Lines (without spatial filter).



Fig. 6 ----- Image of Sine Wave (Using a vertical linear filter in Fourier Transformation Plane.)

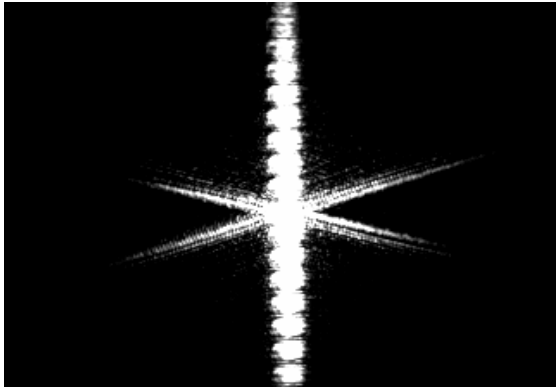


Fig. 7 ----- Experimental Fourier Transformation Configure for Sine Wave with Parallel Lines.

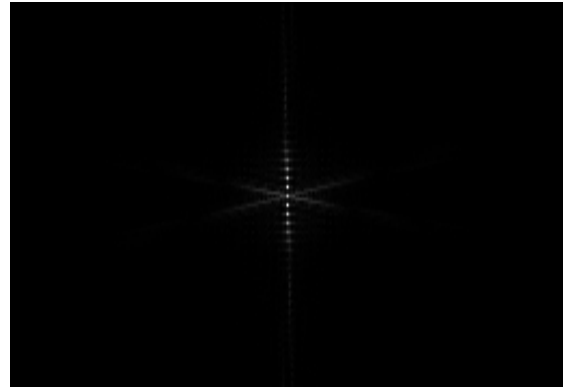


Fig. 8 ----- Spatial Filter (Blocking the vertical column of the points in Transformation Plane.)



Fig. 9 ----- Image of Newton's Head



Fig. 10 ----- Image of Newton's Head manipulation by a vertical linear spatial filter.

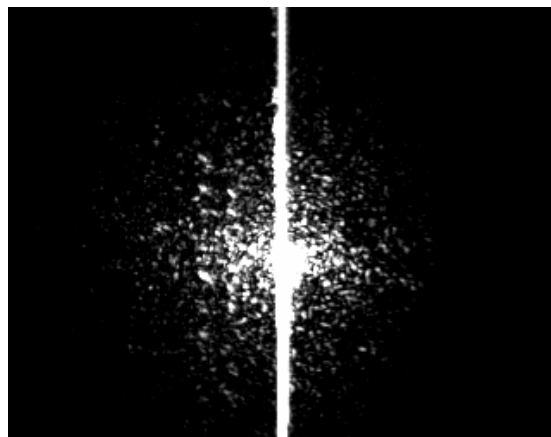


Fig. 11 ----- Fourier Transformation Configuration for Newton' Head.



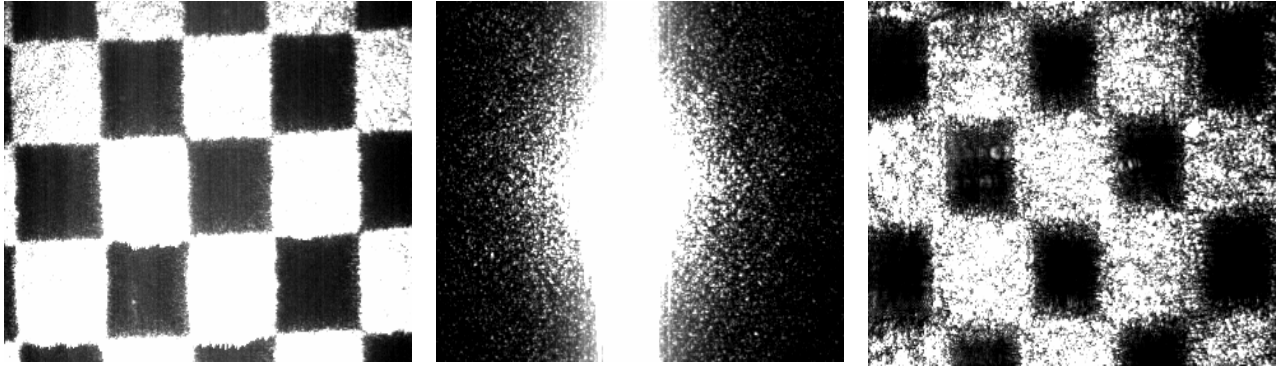


Figure.12 ----- Spatial Filtering for “Black & White Blocks”.

By looking through these pictures, we can see that the difference between Fig. 5 and Fig.6 is: The parallel lines disappear in Fig. 6, however, Fig. 5 has these lines. It is because we use a linear spatial filter in the Transformation Plane to block the transformed configures of the parallel lines in Fig.6, so in the corresponding Image Plane, we can't see these lines. Meanwhile, we also find, the Sine Wave partially diminished because the filter blocks more spatial frequencies in T-Plane (Fig. 2) than we want. If we do the comparison between Fig. 9 and Fig.10, we will find the very similar situation.

Figure. 7 and Figure. 11 are the experimental Fourier Transformation configures for “Sine Wave”(Fig. 5) and “Newton’s Head”(Fig. 9) respectively. And we also can use an image soft ware ----- OPTIMAS 6.5 to do the 2-D Fourier Transformation for images. The following two pictures are computer-based transformation pictures. Figure. 13 is Fourier Transformation for “Sine Wave”, Figure. 14 is Fourier Transformation for “Newton’s Head”. We can they are pretty closed to those we obtained in experiment.

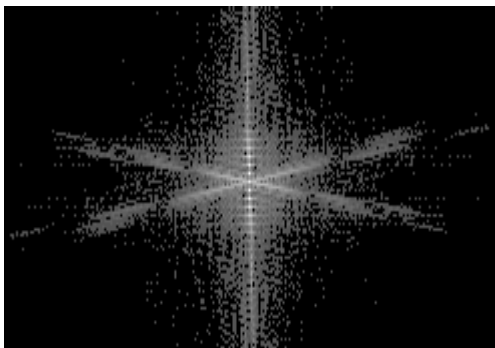


Fig. 13

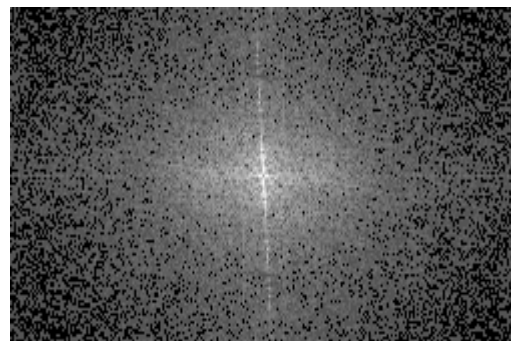


Fig. 14

More ever, we also show the nice Fourier Transformation and Filtering Manipulation for “Orthogonal Symmetric Black-White Grid” in Figure. 12. The left picture is the image, the middle one is the Fourier-Transformation picture, and the right one is the Filtering Image. We also can use OPTIMAS to do the same thing.

This experiment shows a simple application of the theory of Fourier Transform Optics. As we have done, we can change the image of an object by altering its corresponding configurations in the Transformation Domain. Spatial Filter is a most frequently used tool to change the configurations; however, we can also use eraser to erase some points in the transformation plane, or add some new patterns at special points, then we will see the changes in the resulting image.

We also already discussed in “Introduction”, there are some constraints for Fourier Optics. If we want to make it work well, we have to satisfy at least two conditions:

- (1) The focal length of the Fourier Transform Lens  $f$  should be pretty bigger than its transform lens's diameter  $D$ .
- (2) The Size of the object should also be small compared to the focal Length  $f$ .