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# Wilson renormalization group and improved perturbation theory *

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## Abstract

We discuss a resummed perturbation theory based on the Wilson renormalization group. In this formulation the Wilsonian flowing couplings, which generalize the running coupling, enter directly into the loop expansion. In the case of an asymptotically free theory the flowing coupling is well defined since the infrared Landau pole is absent. We show this property in the case of the $\phi_6^3$ theory. We also extend this formulation to the QED theory and we prove that it is consistent with gauge invariance.

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# 1 Introduction

The formulation of quantum field theory based on the Wilson renormalization group [1], which we will call $\Lambda-$RG, studies the evolution in the infrared cutoff $\Lambda$ of the Wilsonian effective action $S(\phi;\Lambda,\Lambda_0)$, where $\Lambda_0$ is some ultraviolet cutoff. This functional is obtained by integrating out all degrees of freedom with momenta higher than $\Lambda$ (and lower than $\Lambda_0$) in the functional integral. By decreasing the scale $\Lambda$ and requiring that physical observables are independent of $\Lambda$ one obtains an evolution equation [2]-[4] for $S(\phi;\Lambda,\Lambda_0)$, which gives a non-perturbative definition of the theory. In this framework one can give simple proofs valid at any order in perturbation theory of many fundamental properties such as renormalizability and infrared finiteness. Moreover the quantum implementation of symmetries is very easily understood [3],[5],[6]. Nevertheless, the practical relevance of this formulation may appear questionable since the $\Lambda-$RG method seems more complex than ordinary renormalization schemes (for instance dimensional regularization). The introduction of a sharp cutoff makes the calculation of general Feynman integrals more difficult. Moreover, in the case of a gauge theory, the cutoffs break explicitly the gauge invariance and one must prove that the theory is invariant once the cutoffs are removed. While the first difficulty is technical and, as we will show in this paper, can be avoided using a suitable cutoff function, the latter is a fundamental issue which must be fulfilled in order to have a consistent theory. In ref. [3, 5] it has been proved that, by appropriately fixing the boundary conditions of the $\Lambda-$RG equations, the effective action of a gauge theory satisfies the Slavnov-Taylor identities at the physical point ($\Lambda = 0$ and $\Lambda_0 \to \infty$). However this proof is inextricably linked to perturbation theory.

The essential advantage of the Wilson formulation is the fact that it provides a non-perturbative definition of the effective action at any scale $\Lambda$ given the action at some (ultraviolet) scale $\Lambda_0$. Unfortunately the $\Lambda-$RG equation corresponds to an infinite system of coupled differential equations for the relevant couplings and the irrelevant vertices of the Wilsonian effective action and its solution needs some approximation. In the last few years there have been several attempts of finding non-perturbative approximate solutions. In general one truncates the space of interactions to few operators according to their dimension and/or uses a derivative expansion [7, 8]. These truncations have been applied expecially to scalar theories and could be very accurate [9]. Similar methods have been applied to gauge theories [10]. In this case one has to face the problem of consistency between truncation and gauge invariance. In general one can truncate the space of interactions in such a way that some of the Slavnov-Taylor identities are satisfied but one can show that the truncation is incompatible with the full set of Slavnov-Taylor identities.

In this paper we consider a recursive approach, first formulated in [11], which mimics perturbation theory analysis, but corresponds to a resummation of higher order of the coupling constant. Some remarks are useful in order to present our idea.

The $\Lambda-$RG equation allows an iterative solution which gives the perturbative expansion. This fact is easily seen if one introduces the cutoff effective action $\Gamma(\phi; \Lambda, \Lambda_0)$, which is related to the Wilsonian action by a Legendre transformation. The evolution equation for the vertices of this functional adds a loop, thus the vertices at loop $\ell$ are determined by the vertices at lower loops. Therefore, from the effective action at zero loop, *i.e.* the classical action, one can determine this functional at any loop order.

The improved formulation is similar: a finite number of $\Lambda-$dependent couplings, the flowing couplings, is sorted out. These couplings correspond at $\Lambda = 0$ to the physical couplings and are computable at any $\Lambda$ solving a finite set of differential equations. The remaining part of the cutoff effective action is obtained using recursive integral equations. In this way the renormalized coupling constant is replaced in the loop integrals by the flowing coupling at the scale given by the loop momenta (at least in the case of a sharp cutoff). This point of view is very close to the resummed perturbation theory, in which higher order corrections reconstruct the running coupling constant $g(q^2)$, where $q$ is the loop momentum [12]. This resummation is well established in the large $N_f$ limit of QED, and it is applied as an ansatz (naive non-Abelianization procedure) to the QCD [13]. In the latter case the one-loop running coupling constant diverges at the infrared Landau pole, and the integration over the low momenta becomes ambiguous [14]. Our improved formulation is systematic, *i.e.* applies equally well to the non-asymptotically free and asymptotically free theories. In the latter case the infrared Landau pole is absent since the flowing coupling remains finite in all range of the momenta. In this paper we show this property in the case of the one-loop improved $\phi_6^3$ theory.

For a gauge theory it is crucial that the improved perturbation theory does not produce a breaking of (quantum) gauge invariance, *i.e.* one has to show that the solution of the $\Lambda-$RG equation satisfies at $\Lambda = 0$ the Slavnov-Taylor identities. In this paper we consider the case of QED and we show explicitly that the one-loop improved solution satisfies the Ward identities up to negative powers of the ultraviolet cutoff $\Lambda_0$. As in the standard resummed perturbation theory, the presence of the Landau pole in the ultraviolet region implies that one cannot remove $\Lambda_0$ and therefore the Ward identities are recovered only for momenta much lower than $\Lambda_0$.

The paper is organized as follows. In sections 2 and 3 we recall the details of the $\Lambda-$RG formulation for the massless $\phi_4^4$ theory in the perturbative and in the improved case, respectively. In section 4 we analyze the massive $\phi_6^3$ theory as a pedagogical example of asymptotically free theory. In section 5 we present the improved perturbation theory for QED and we compute the flowing couplings at one loop. In section 6 we prove that the one-loop improved formulation is consistent with gauge invariance. In section 7 we compare our approach with the standard resummed perturbation theory and section 8 contains some conclusions. The choice of the cutoff function and the conventions are described in two appendices.

## 2 Remarks on perturbative $\Lambda-$RG method

In order to fix the notations, we review the usual (non-improved) $\Lambda-$RG formulation in the case of massless $\phi^4$ theory in four dimensions [4]. The starting point is the evolution equation for the cutoff effective action $\Gamma(\phi;\Lambda,\Lambda_0)$

$$\Lambda\partial_\Lambda(\Gamma(\phi;\Lambda,\Lambda_0)-\frac{1}{2}\phi\cdot\Delta^{-1}_{\Lambda\Lambda_0}\phi)=\hbar I(\phi;\Lambda,\Lambda_0) \tag{1}$$

where

$$\phi\cdot\Delta^{-1}_{\Lambda\Lambda_0}\phi\equiv\int_q\phi(-q)\Delta^{-1}_{\Lambda\Lambda_0}(q)\phi(q),\qquad \int_q\equiv\int\frac{d^4q}{(2\pi)^4}$$

and

$$I(\phi;\Lambda,\Lambda_0)=-\frac{1}{2}\int_q\Lambda\partial_\Lambda\Delta^{-1}_{\Lambda\Lambda_0}(q)\Gamma_2^{-1}(q;\Lambda,\Lambda_0)\bar{\Gamma}_{\phi\phi}(q,-q;\phi;\Lambda,\Lambda_0)\Gamma_2^{-1}(q;\Lambda,\Lambda_0).$$

The cutoff propagator $\Delta_{\Lambda\Lambda_0}(q)$ is obtained by multiplying the free propagator $\Delta(q)\equiv 1/q^2$ with the cutoff function $K_{\Lambda\Lambda_0}(q)$. This function cuts the frequencies below the infrared cutoff $\Lambda$ and above the ultraviolet cutoff $\Lambda_0$. The auxiliary functional $\bar{\Gamma}_{\phi\phi}$ depends non-linearly on the cutoff effective action and it is defined in [4]. The physical effective action $\Gamma(\phi)$ is extracted from $\Gamma(\phi;\Lambda,\Lambda_0)$ performing the limit $\Lambda\to 0$ and $\Lambda_0\to\infty$. It is also convenient to introduce the functional

$$\Pi(\phi;\Lambda,\Lambda_0)\equiv\Gamma(\phi;\Lambda,\Lambda_0)-\frac{1}{2}\phi\cdot(\Delta^{-1}_{\Lambda\Lambda_0}-\Delta^{-1})\phi$$

which at the tree level coincides with $S_{cl}(\phi)$. At higher orders the cutoff vertices $\Pi_{2n}(p_i;\Lambda,\Lambda_0)$ and $\Gamma_{2n}(p_i;\Lambda,\Lambda_0)$ are equal and become the physical vertices in the limit $\Lambda\to 0$ and $\Lambda_0\to\infty$. The evolution equation (1) can be iteratively solved by using the loop expansion $\Pi^{[\ell]}=\Pi^{(0)}+\hbar\Pi^{(1)}+\ldots+\hbar^\ell\Pi^{(\ell)}$. One obtains

$$\Pi^{(\ell)}(\phi;\Lambda,\Lambda_0)=\hbar\int_\Lambda^{\Lambda_0}\frac{d\lambda}{\lambda}I^{(\ell-1)}(\phi;\lambda,\Lambda_0)+\text{boundary conditions.}$$

In order to specify the boundary conditions the effective action $\Pi(\phi;\Lambda,\Lambda_0)$ is split into a relevant part

$$\Pi_{rel}(\phi;\Lambda,\Lambda_0)=\int_x\frac{1}{2}Z(\Lambda)\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}\sigma_m(\Lambda)\phi^2+\frac{1}{4!}\sigma_g(\Lambda)\phi^4$$

and an irrelevant part $\Pi_{irr}=\Pi-\Pi_{rel}$ (the same decomposition also holds for the functional $I(\phi;\Lambda,\Lambda_0)$). The relevant couplings are given by

$$Z(\Lambda)=\partial_{p^2}\Pi_2|_{p^2=\mu^2}\,,\quad \sigma_m(\Lambda)=\Pi_2|_{p^2=0}\,,\quad \sigma_g(\Lambda)=\Pi_4|_{p_i=\bar{p}_i}$$

where $\bar{p}_i$ is the symmetric point, defined by $\bar{p}_i \cdot \bar{p}_j = \frac{\mu^2}{3}(4\delta_{ij} - 1)$. The boundary conditions for the relevant couplings are fixed at any order at the physical scale $\Lambda = 0$ by

$$Z^{(\ell)}(0) = \delta_{\ell 0}, \quad \sigma_m^{(\ell)}(0) = 0, \quad \sigma_g^{(\ell)}(0) = g\delta_{\ell 0}.$$

The boundary conditions for the irrelevant part are fixed at the ultraviolet scale $\Lambda = \Lambda_0$ and are trivial:

$$\Pi_{irr}^{(\ell)}\Big|_{\Lambda=\Lambda_0} = 0.$$

With these boundary conditions the recursive solution of the evolution equation exists at any perturbative order in the physical limit $\Lambda_0 \to \infty$ and $\Lambda \to 0$ for non-exceptional configurations of external momenta [4].

# 3 Improved perturbation theory

In this section we formulate more precisely our improved perturbation theory for the massless $\phi^4$ theory in four dimensions [11]. We introduce the rescaled vertices

$$\hat{\Pi}_{2n}(p_i; \Lambda, \Lambda_0) = Z^{-n}(\Lambda)\Pi_{2n}(p_i; \Lambda, \Lambda_0)$$

which satisfy the following evolution equation

$$(\Lambda\partial_\Lambda + n\frac{\dot{Z}}{Z})\hat{\Pi}_{2n}(p_i; \Lambda, \Lambda_0) = \hat{I}_{2n}(p_i; \Lambda, \Lambda_0), \tag{2}$$

where the dot denotes the $\Lambda\partial_\Lambda$ derivative and the $\hat{I}_{2n}(p_i; \Lambda, \Lambda_0)$ are the vertices of the functional $I(\phi; \Lambda, \Lambda_0)$ after the rescaling. In particular one has

$$\hat{I}_2(p; \Lambda, \Lambda_0) = -\frac{1}{2}\int_q \hat{M}(q; \Lambda, \Lambda_0)\hat{\Pi}_4(q, p, -p, -q; \Lambda, \Lambda_0)$$

and

$$\hat{I}_4(p_1, \dots, p_4; \Lambda, \Lambda_0) = -\frac{1}{2}\int_q \hat{M}(q; \Lambda, \Lambda_0)[\hat{\Pi}_6(q, p_1 \dots p_4, -q; \Lambda, \Lambda_0)$$
$$-\sum_P \hat{\Pi}_4(q, p_{i_1}, p_{i_2}, -Q; \Lambda, \Lambda_0)\hat{\Gamma}_2^{-1}(Q; \Lambda, \Lambda_0)\hat{\Pi}_4(Q, p_{i_3}, p_{i_4}, -q; \Lambda, \Lambda_0)],$$

where $Q = q + p_{i_1} + p_{i_2}$, the sum is over six permutations and the measure $\hat{M}$ is given by

$$\hat{M}(q; \Lambda, \Lambda_0) \equiv \hat{\Gamma}_2^{-1}(q; \Lambda, \Lambda_0)\Lambda\partial_\Lambda\Delta_{\Lambda\Lambda_0}^{-1}(q)\hat{\Gamma}_2^{-1}(q; \Lambda, \Lambda_0). \tag{3}$$

To calculate the improved vertices we use an iterative procedure starting from the improved tree level cutoff action

$$\hat{\Gamma}^{(0)} = \frac{1}{2}\hat{\phi} \cdot \Delta_{\Lambda\Lambda_0}^{-1}\hat{\phi} + \frac{1}{4!}\int_x \hat{g}(\Lambda)\hat{\phi}^4. \tag{4}$$

The improved perturbation theory consists in solving the evolution equations as for the usual perturbative expansion but in terms of $\hat{g}(\Lambda)$. This coupling will be obtained at the end of the iterative procedure by solving its evolution equation.

The iteration starts by inserting (4) in the r.h.s. of (2). In this way one obtains the one-loop relevant coupling $\hat{\Pi}_2^{[1]}|_{p=0}$ and the irrelevant vertices $\hat{\Pi}_{2n,irr}^{[1]}$ in terms of an integral in $\lambda$ which involve the flowing coupling $\hat{g}(\lambda)$ and the rescaling function $Z(\lambda)$

$$\hat{\Pi}_2^{[1]}(0;\Lambda,\Lambda_0) = Z^{-1}(\Lambda)\int_0^{\Lambda}\frac{d\lambda}{\lambda}Z(\lambda)\hat{I}_2^{[0]}(0;\lambda,\Lambda_0),$$

$$\hat{\Pi}_{2n,irr}^{[1]}(p_i;\Lambda,\Lambda_0) = -Z^{-n}(\Lambda)\int_{\Lambda}^{\Lambda_0}\frac{d\lambda}{\lambda}Z^n(\lambda)\hat{I}_{2n,irr}^{[0]}(p_i;\lambda,\Lambda_0). \qquad (5)$$

From these vertices one obtains the $\hat{I}_{2n}^{[1]}(p_i;\Lambda,\Lambda_0)$ and constructs the second iteration. After iterating this procedure $\ell$ times, $\hat{\Pi}_2^{[\ell]}|_{p=0}$ and $\hat{\Pi}_{2n,irr}^{[\ell]}$ are given in terms of multiple integrals, over the various scales $\lambda_i$ generated by the iteration, of complicated expressions involving the flowing coupling and the rescaling function at the various scales $\lambda_i$. In order to obtain the functions $Z(\Lambda)$ and $\hat{g}(\Lambda)$ needed to compute these integrals, one uses the definitions $\partial_{p^2}\hat{\Pi}_2^{[\ell]}|_{p^2=\mu^2} \equiv 1$ and $\hat{g}(\Lambda) = \hat{\Pi}_4^{[\ell]}\Big|_{p_i=\bar{p}_i}$ which give the evolution equations for $Z(\Lambda)$ and $\hat{g}(\Lambda)$ at order $\ell$

$$\frac{\dot{Z}}{Z} = \partial_{p^2}\hat{I}_2^{[\ell-1]}\Big|_{p^2=\mu^2}, \qquad (6)$$

$$\Lambda\partial_\Lambda\hat{g} + 2\frac{\dot{Z}}{Z}\hat{g} = \hat{I}_4^{[\ell-1]}\Big|_{p_i=\bar{p}_i}. \qquad (7)$$

These equations are solved with the boundary conditions $Z(0)=1$ and $\hat{g}(0)=g(\mu)$, where $g(\mu)$ is the coupling constant evaluated at the subtraction point $\mu$. In general the r.h.s. of (6) and (7) are functionals of $\hat{g}(\lambda_i)$ and $Z(\lambda_i)$ thus one has complicated integro-differential equations.

In this paper we perform the calculation only at the first order, where (6) and (7) are simple differential equations, which in general can be solved analytically. By inserting the solution of these equations in (5) and setting $\Lambda = 0$ one can explicitly compute the one-loop improved physical vertices. They are given by the same expression of the one-loop perturbative vertices with the coupling replaced by the flowing coupling $\hat{g}(\lambda)$. This is similar to the standard improved theory in which one substitutes the coupling with the running coupling $g(q^2)$ in the Feynman diagrams. The relation between the two approaches is discussed in section 7.

### 3.1 Explicit 1-loop calculations

The rescaling function in the one-loop improved $\phi^4$ theory is trivial because $\dot{Z}/Z = \partial_{p^2}\hat{I}_2^{(0)}\Big|_{\mu} \equiv 0$, and therefore $Z(\Lambda)=1$ for any $\Lambda$. In the $\Lambda_0\to\infty$ limit the 1-loop flowing coupling satisfies the evolution equation

$$\Lambda\partial_\Lambda \hat{g}(\Lambda) = \hat{I}_4^{[0]}\Big|_{p_i=\bar{p}_i} = \frac{3}{16\pi^2}\hat{g}^2(\Lambda)F(\Lambda^2/\mu^2), \tag{8}$$

where[2]

$$F(\Lambda^2/\mu^2) = -16\pi^2 \int_q \dot{\Delta}_{\Lambda\infty}(q)\Delta_{\Lambda\infty}(q+\bar{p}), \quad \bar{p}^2 = \mu^2 \tag{9}$$

can be exactly calculate specifying the cutoff function. Notice that $F(\Lambda^2/\mu^2)$ vanishes for $\Lambda = 0$ and $F \to 1$ for $\Lambda \to \infty$ for any choice of the cutoff function (see appendix A). Using the power-law cutoff function given in appendix A (see equation (31)) one finds

$$F(\Lambda^2/\mu^2) = \frac{2\Lambda^2(6\Lambda^4 + 7\Lambda^2\mu^2 + \mu^4)}{\mu^2(\mu^2 + 4\Lambda^2)^2} - \frac{48\Lambda^8 \mathrm{ArcTanh}\sqrt{\mu^2/(\mu^2+4\Lambda^2)}}{\sqrt{\mu^6}(\mu^2+4\Lambda^2)^{5/2}}, \tag{10}$$

with asymptotic limits

$$F(\Lambda^2/\mu^2) = 2\frac{\Lambda^2}{\mu^2} - 2\frac{\Lambda^4}{\mu^4} \quad (\Lambda << \mu), \quad F(\Lambda^2/\mu^2) = 1 - \frac{2}{5}\frac{\mu^2}{\Lambda^2} + \frac{1}{7}\frac{\mu^4}{\Lambda^4} \quad (\Lambda >> \mu).$$

The solution of equation (8)

$$\hat{g}(\Lambda) = \frac{g(\mu)}{1 - \frac{3}{16\pi^2}g(\mu)\int_0^\Lambda \frac{d\lambda}{\lambda}F(\lambda^2/\mu^2)}, \quad \hat{g}(0) = g(\mu) \tag{11}$$

can be expressed in terms of elementary functions but, for sake of simplicity, we do not report the lengthy formula. In figure 1 we show $\hat{g}(\Lambda)$ as a function of $\Lambda/\mu$.

In a previous article [11] we calculated the flowing coupling with the sharp cutoff function $K_{\Lambda\infty}(q) = \theta(q^2/\Lambda^2 - 1)$ . [3] In both cases the flowing coupling has a Landau pole, *i.e.* the denominator of (11) vanishes at finite $\Lambda = \Lambda_L$. The existence of the pole is a general fact because for $\Lambda \geq \bar{\Lambda} >> \mu$ equation (8) has the *universal* ( *i.e.* independent on the cutoff function) asymptotic solution

$$\hat{g}_{as}(\Lambda) = \frac{\hat{g}(\bar{\Lambda})}{1 - \frac{3}{16\pi^2}\hat{g}(\bar{\Lambda})\log\Lambda/\bar{\Lambda}}.$$

However, the exact position of the Landau pole depends on the value of $g(\mu)$ and on the choice of the cutoff function. For instance, by fixing $g(\mu) = 4\pi$, with the power-law cutoff one has $\Lambda_L = 43.47\mu$ while with the sharp cutoff one has $\Lambda_L = 39.79\mu$. We have also computed the pole position in the case of the exponential cutoff $K_{\Lambda\infty}(q) = 1 - \exp(-q^2/\Lambda^2)$ obtaining $\Lambda_L = 37.74\mu$.

[2]In the r.h.s. of (8) we can take the $\Lambda_0 \to \infty$ limit because the $\Lambda\partial_\Lambda$−derivative cuts the higher momenta.

[3]With the sharp cutoff we have been able to evaluate exactly $F(\Lambda^2/\mu^2)$ but not $\hat{g}(\Lambda)$ .

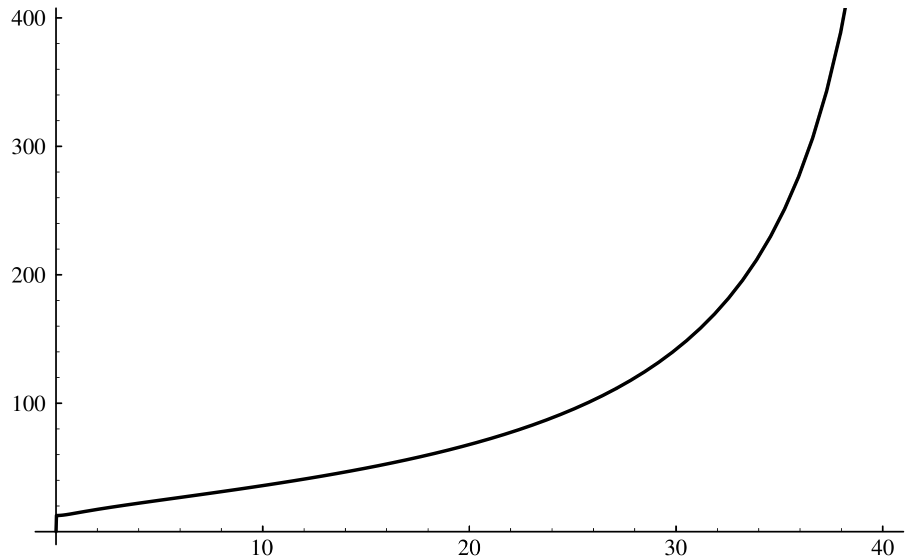


Figure 1: Wilsonian flowing coupling $\hat{g}$ as function of $\Lambda/\mu$ calculated with the power-law cutoff function. The initial point is $\hat{g}(0) = g = 4\pi$, the Landau pole is at $\Lambda = 43.47\mu$.

By inserting in (5) the flowing coupling calculated from (11) and taking $\Lambda = 0$ one obtains the physical one-loop improved vertices. Notice that the presence of the Landau pole at $\Lambda = \Lambda_L$ implies that the ultraviolet cutoff $\Lambda_0$ cannot be removed in this case, although the theory is perturbatively renormalizable. This corresponds to the property of triviality, entailing that the $\Lambda_0 \to \infty$ limit is possible only if the coupling $g(\mu)$ vanishes.

## 4 $\phi_6^3$ theory

The rescaling function $Z(\Lambda)$ was not involved in the previous discussion on the $\phi_4^4$ theory at one loop. Therefore, before treating QED, it is useful to consider the (massive) $\phi_6^3$ theory in six dimension as an example in which $Z(\Lambda)$ is not trivial at one loop. This theory is interesting even because it mimics some features of QCD ( *i.e.* it is an asymptotically free theory). Moreover, we analyze the presence of a fixed mass $m$ different from zero.

The improved tree level cutoff action is

$$\hat{\Gamma}^{(0)} = \frac{1}{2}\hat{\phi}\cdot\Delta^{-1}_{\Lambda\Lambda_0}\hat{\phi} + \frac{1}{3!}\int_x \hat{g}(\Lambda)\hat{\phi}^3, \quad \Delta_{\Lambda\Lambda_0}(q) \equiv \frac{K_{\Lambda\Lambda_0}(q)}{q^2+m^2}$$

and the one-loop improved evolution equations in the $\Lambda_0 \to \infty$ limit read

$$\Lambda\partial_\Lambda\hat{\Pi}_2 + \frac{\dot{Z}}{Z}\hat{\Pi}_2 = -\hat{g}^2(\Lambda)\int_q \dot{\Delta}_{\Lambda\infty}(q)\Delta_{\Lambda\infty}(q+p),$$

$$\Lambda\partial_\Lambda\hat{\Pi}_3 + \frac{3}{2}\frac{\dot{Z}}{Z}\hat{\Pi}_3 = 3\hat{g}^3(\Lambda)\int_q \dot{\Delta}_{\Lambda\infty}(q)\Delta_{\Lambda\infty}(q+p)\Delta_{\Lambda\infty}(q+p+p')$$

and similarly for other vertices[4]. From the definitions $\partial_{p^2}\hat{\Pi}_2\Big|_{p=0} \equiv 1$ and $\hat{g} \equiv \hat{\Pi}_3\Big|_{p_i=0}$ one has

$$\frac{\dot{Z}}{Z} = -\frac{1}{6}r\hat{g}^2F_\phi, \quad \Lambda\partial_\Lambda\hat{g} = -r\hat{g}^3F_g - \frac{3}{2}\frac{\dot{Z}}{Z}\hat{g}, \quad r = \frac{1}{(4\pi)^3} \tag{12}$$

where

$$\begin{aligned} F_\phi(\Lambda) &= 6(4\pi)^3\,\partial_{p^2}\textstyle\int_q \dot{\Delta}_{\Lambda\infty}(q)\Delta_{\Lambda\infty}(q+p)\Big|_{p=0}, \\ F_g(\Lambda) &= -3(4\pi)^3\textstyle\int_q \dot{\Delta}_{\Lambda\infty}(q)\Delta_{\Lambda\infty}(q)^2 \end{aligned} \tag{13}$$

are functions growing from 0 to 1 which can be exactly calculated specifying the cutoff function. From (12) one finds that the flowing coupling $\hat{g}$ is determined by the differential equation

[4] We do not consider the $n = 1$ vertex because it is momentum independent and then vanishes by zero-momentum subtraction.

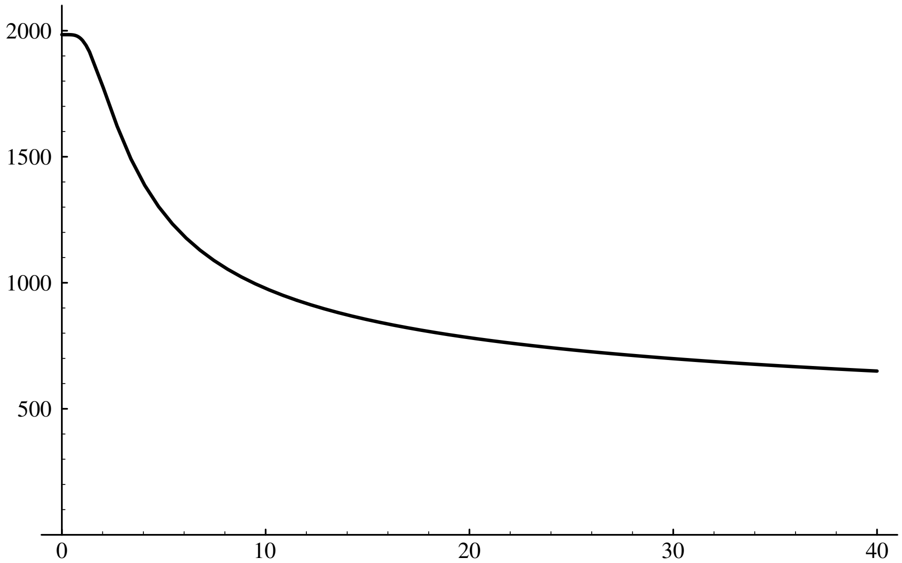


Figure 2: Wilsonian flowing coupling $\hat{g}^2$ as function of $\Lambda/m$ in $\phi_6^3$ theory calculated using the power-law cutoff function. The initial point is $\hat{g}^2(0) = g^2 = (4\pi)^3$.

$$\Lambda\partial_\Lambda \hat g = [-F_g(\Lambda) + \frac{1}{4}F_\phi(\Lambda)] r\hat g^3, \qquad \hat g(0) = g. \tag{14}$$

We solved this equation using the functions $F_\phi(\Lambda)$ and $F_g(\Lambda)$ computed with the power-law cutoff (see appendix A) and the result is displayed in figure 2. In the ultraviolet limit $\Lambda \geq \bar\Lambda >> m$ equation (14) becomes

$$\Lambda\partial_\Lambda \hat g_{as} = -\frac{3}{4} r\hat g_{as}^3 = -b_1 \hat g_{as}^3, \qquad b_1 = \frac{3}{256\pi^3}$$

and therefore the asymptotic flowing coupling has the same functional form of the running coupling

$$g_{as}^2(\Lambda) = \frac{\bar g^2}{1 + b_1 \bar g^2 \log \Lambda/\bar\Lambda}, \qquad g_{as}(\bar\Lambda) = \bar g.$$

Notice that the asymptotic coupling is ill defined in the infrared for $\Lambda = \Lambda_L = \bar\Lambda e^{-1/(b_1 \bar g^2)}$ (Landau pole), on the contrary the flowing $\hat g(\Lambda)$ is regular for any $\Lambda$. From (12) one also obtains the rescaling function

$$Z(\Lambda) = \exp[-\frac{1}{6} r \int_0^\Lambda \frac{d\lambda}{\lambda} \hat g^2(\lambda) F_\phi(\lambda)]$$

which is well defined for any $\Lambda$ and decreases to zero for $\Lambda \to \infty$. In the asymptotic limit one has $Z(\Lambda) \to \left(1 + b_1 \bar g^2 \log \Lambda/\bar\Lambda\right)^{-\frac{r}{6b_1}} Z(\bar\Lambda)$.

# 5 Improved QED

In this section we generalize the previous calculations to the QED case. The crucial point is to show that the improved theory at $\Lambda = 0$ satisfies the Ward identities associated with the gauge transformation

$$\delta_\varepsilon \psi(x) = ie\varepsilon(x)\psi(x), \quad \delta_\varepsilon \bar\psi(x) = -ie\bar\psi(x)\varepsilon(x), \quad \delta_\varepsilon A_\mu(x) = \partial_\mu \varepsilon(x).$$

The tree level improved cutoff action is

$$\hat\Gamma^{(0)}(\hat\phi, \hat a, \hat e; \Lambda, \Lambda_0) = \int_x \frac{1}{2} \hat A_\mu (D^{-1}_{\Lambda\Lambda_0})^{\mu\nu} \hat A_\nu + \hat{\bar\psi} S^{-1}_{\Lambda\Lambda_0} \hat\psi + \hat e(\Lambda) \hat{\bar\psi} \hat{\not A} \hat\psi, \tag{15}$$

where $\hat e(\Lambda)$ is the flowing coupling and we have introduced the rescaled fields $\hat\psi = Z_\psi^{1/2}\psi$, $\hat{\bar\psi} = Z_\psi^{1/2}\bar\psi$ and $A_\mu = Z_A^{1/2} A_\mu$. The cutoff propagators are

$$D_{\Lambda\Lambda_0,\mu\nu}(k) = -\left(\frac{g_{\mu\nu}}{k^2} - (1 - \hat a(\Lambda))\frac{k_\mu k_\nu}{k^4}\right) K_{\Lambda\Lambda_0}(k)$$

and

$$S_{\Lambda\Lambda_0}(p) = -\frac{\not p + m}{p^2 - m^2} K_{\Lambda\Lambda_0}(p),$$

where $K_{\Lambda\Lambda_0}(k)$ and $K_{\Lambda\Lambda_0}(p)$, after analytic continuation in euclidean space, become the power-law cutoff functions defined in appendix A. In particular for the photon propagator we use the cutoff function (31) while for the electron propagator we use the massive cutoff function (32). Notice that the photon propagator contains the $\Lambda-$dependent gauge fixing parameter $\hat a(\Lambda)$. As we will see, this is required by gauge invariance.

The evolution equations for the rescaled vertices are obtained following the same steps discussed in Section 3. In this case the measure $\hat M$ (corresponding to (3)) contains both the electron and photon propagators. In the following we use the notation $\Lambda\partial_\Lambda D_{\Lambda\Lambda_0,\mu\nu}$ for the contributions to $\hat M$ coming from the photon propagator even though the derivative acts only on the cutoff function and not on the gauge fixing parameter $\hat a(\Lambda)$.

Starting from (15) one can iteratively construct the improved vertex functions at higher orders. The rescaling functions are obtained solving the corresponding evolution equations with boundary conditions $Z_A(0) = 1,\ Z_\psi(0) = 1$, while $\hat e(\Lambda)$ and $\hat a(\Lambda)$ are not independent functions. Indeed gauge invariance requires the flowing coupling $\hat e(\Lambda)$ to be related to the rescaling function $Z_A(\Lambda)$ by

$$\hat e(\Lambda) = e Z_A^{-1/2}(\Lambda). \tag{16}$$

Similarly the flowing gauge fixing coupling $\hat a(\Lambda)$ must satisfy the relation $\hat a(\Lambda) = a Z_A(\Lambda)$, where $a$ is a fixed number which does not affect the physics (for instance $a = 1$ in the Feynman gauge). In this way the tree level improved action in terms of the non-rescaled fields

$$\Pi^{(0)} = \int_x -Z_A(\Lambda)[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\hat a(\Lambda)}(\partial\cdot A)^2] + Z_\psi(\Lambda)\bar\psi(i\not\partial + \hat e Z_A^{1/2}\not A - m)\psi$$

satisfies the standard Ward identity

$$W(x)\Pi^{(0)} = (-\partial_\mu\frac{\delta\Pi^{(0)}}{\delta A_\mu} - ie\bar\psi\frac{\delta\Pi^{(0)}}{\delta\bar\psi} + ie\frac{\delta\Pi^{(0)}}{\delta\psi}\psi) = \frac{\Box}{a}\partial\cdot A(x)$$

at any scale $\Lambda$. Through this property we will prove that the physical improved action $\Pi^{[1]}(\phi; 0, \Lambda_0)$ satisfies Ward identities up order $\mathcal{O}(1/\Lambda_0)$, then in the limit $\Lambda_0 \to \infty$ "gauge-invariance" is preserved by the improved perturbative expansion, at least at one loop.

We now perform some explicit computations. We denote by $\hat\Pi^{[1]}_{\mu_1\ldots\mu_n\alpha_1\ldots\alpha_{2m}}(p_i; \Lambda, \Lambda_0)$ the one-loop improved vertices with $n$ photons and $m$ pairs of fermions.

The (inverse) photon propagator evolution equation is

$$(\Lambda\partial_\Lambda + \frac{\dot{Z}_A}{Z_A})\hat{\Pi}^{[1]}_{\mu\nu}(p;\Lambda,\Lambda_0) = \hat{e}^2(\Lambda)\hat{I}_{\mu\nu}(p;\Lambda,\Lambda_0), \tag{17}$$

where

$$\hat{I}_{\mu\nu}(p;\Lambda,\Lambda_0) = i\int_q \Lambda\partial_\Lambda \mathrm{Tr}(\gamma_\mu S_{\Lambda\Lambda_0}(q)\gamma_\nu S_{\Lambda\Lambda_0}(q+p))$$

and we have explicitly written the dependence on $\hat{e}(\Lambda)$. From this equation and the normalization condition $-\,\partial_{p^2}\frac{1}{3}t^{\mu\nu}\hat{\Pi}_{\mu\nu}\Big|_{p^2=0} = 1$, where $t^{\mu\nu} = g^{\mu\nu} - p^\mu p^\nu/p^2$, one obtains the evolution equation for the rescaling function $Z_A(\Lambda)$. In particular choosing the power-law cutoff function and taking the limit $\Lambda_0 \to \infty$ one has

$$\frac{\dot{Z}_A}{Z_A} = -\hat{e}^2(\Lambda)\,\partial_{p^2}\frac{1}{3}t^{\mu\nu}\hat{I}_{\mu\nu}\Big|_{p^2=0} = -\frac{\hat{e}^2(\Lambda)}{6\pi^2}\frac{\Lambda^4(5\Lambda^4+14\Lambda^2m^2+6m^4)}{5(\Lambda^2+m^2)^4}.$$

Using (16) and the boundary condition $Z(0) = 1$ one finds

$$Z_A(\Lambda) = 1 - \frac{e^2}{6\pi^2}\left[\frac{1}{2}\log\frac{\Lambda^2+m^2}{m^2} - \frac{\Lambda^2(7\Lambda^4+19\Lambda^2m^2+10m^4)}{20(\Lambda^2+m^2)^3}\right].$$

In figures 3 and 4 we display the flowing couplings $\hat{\alpha}(\Lambda) = \frac{\hat{e}^2(\Lambda)}{4\pi}$ and $\hat{a}(\Lambda) = aZ_A(\Lambda)$ calculated with this choice of the cutoff function. Notice that the coupling $\hat{\alpha}(\Lambda)$ goes to infinity for a finite value $\Lambda = \Lambda_L$, the Landau pole. The exact position of the pole depends on the choice of $K_{\Lambda\infty}(q)$.

The electron rescaling function can be obtained from the (inverse) electron propagator evolution equation

$$(\Lambda\partial_\Lambda + \frac{\dot{Z}_\psi}{Z_\psi})\hat{\Pi}^{[1]}_{\alpha\beta}(p;\Lambda,\Lambda_0) = \hat{e}^2(\Lambda)\hat{I}_{\alpha\beta}(p;\Lambda,\Lambda_0), \tag{18}$$

where

$$\hat{I}_{\alpha\beta}(p;\Lambda,\Lambda_0) = -i\int_q \Lambda\partial_\Lambda\left[\gamma_\mu S_{\Lambda\Lambda_0}(q+p)\gamma_\nu D^{\mu\nu}_{\Lambda\Lambda_0}(q)\right]_{\alpha\beta},$$

and the normalization condition $\partial_{p^\mu}\hat{\Pi}_{\alpha\beta}\Big|_{p^2=0} = \gamma^\mu_{\alpha\beta}$. Also in this case, using the power-law cutoff function, one can compute explicitly $Z_\psi(\Lambda)$. Here we give only its asymptotic limit for $\Lambda_L >> \Lambda >> m$

$$Z_\psi(\Lambda) \simeq 1 - \frac{1}{16\pi^2}ae^2\log\frac{\Lambda^2}{m^2}.$$

For further references we report the vertex evolution equation

$$(\Lambda\partial_\Lambda + \frac{\dot{Z}_\psi}{Z_\psi} + \frac{1}{2}\frac{\dot{Z}_A}{Z_A})\Pi^{[1]}_{\mu\alpha\beta}(p,p';\Lambda,\Lambda_0) = \hat{e}^3(\Lambda)I_{\mu\alpha\beta}(p,p';\Lambda,\Lambda_0), \tag{19}$$

where

$$I_{\mu\alpha\beta}(p,p';\Lambda,\Lambda_0) = -i\int_q \Lambda\partial_\Lambda\left[\gamma^\rho S_{\Lambda\Lambda_0}(q+p)\gamma_\mu S_{\Lambda\Lambda_0}(q+p')\gamma^\sigma D_{\Lambda\Lambda_0,\sigma\rho}(q)\right]_{\alpha\beta}.$$

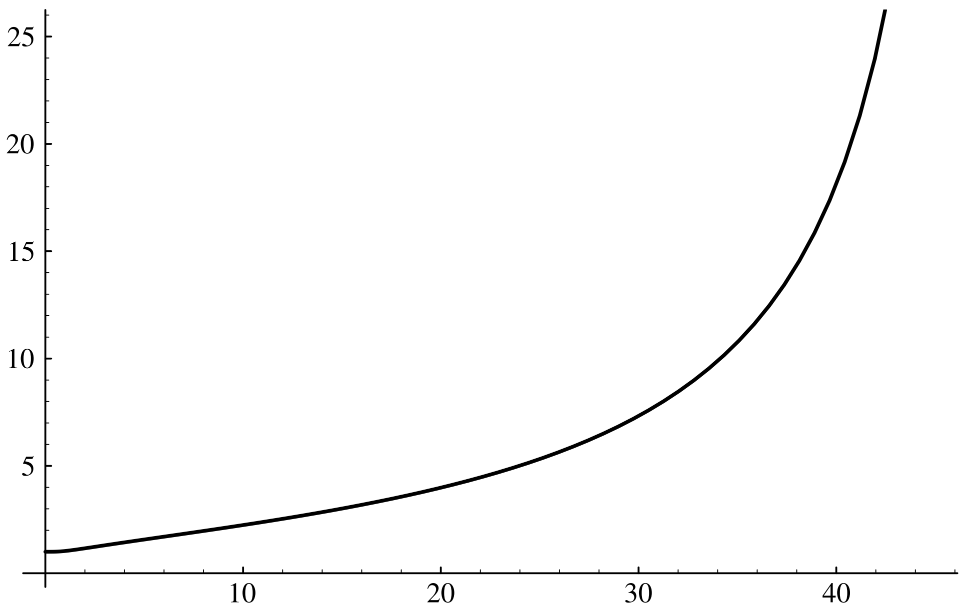


Figure 3: Wilsonian flowing coupling $\hat{\alpha}$ as function of $\Lambda/m$ in QED calculated using the power-law cutoff function. The initial point is $\hat{\alpha}(0) = \alpha = 1$.

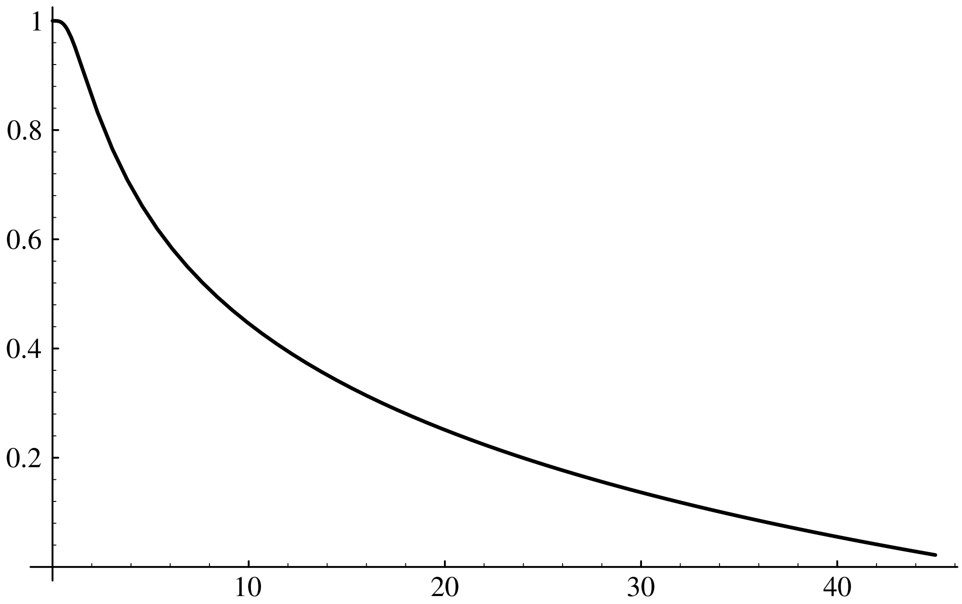


Figure 4: Wilsonian flowing coupling $\hat{a}$ as function of $\Lambda/m$. The initial point is $\hat{a}(0) = a = 1$ (Feynman gauge).

# 6 Gauge invariance of the improved theory

In this section, we explicitly check the gauge invariance of the one-loop improved vertices at $\Lambda = 0$. In particular we analyze the transversality of the photon propagator and the Ward identity for the vertex. The longitudinal part of the photon propagator $\Pi_L \equiv \frac{p^\mu p^\nu}{p^2}\Pi_{\mu\nu}$ is obtained from (17) and it is given by

$$\Pi_L^{[1]}(p;0,\Lambda_0) = -\int_0^{\Lambda_0} \frac{d\lambda}{\lambda} Z_A(\lambda)\hat{e}^2(\lambda)\hat{I}_{L,irr}(p;\lambda), \tag{20}$$

where

$$\hat{I}_{L,irr}(p;\lambda) = \hat{I}_L - \hat{I}_T\Big|_{p=0} - p^2\, \partial_{\bar{p}^2}\hat{I}_L\Big|_{\bar{p}^2=0}, \quad \hat{I}_L = \frac{p^\mu p^\nu}{p^2}\hat{I}_{\mu\nu}, \quad \hat{I}_T = \frac{1}{3}t^{\mu\nu}\hat{I}_{\mu\nu} \tag{21}$$

(here and in the following the $\Lambda_0-$dependence in $\hat{I}_L$ is understood). The subtractions in the integrand of (20) are a consequence of isolating the relevant couplings in $\Pi_{\mu\nu}$ and of the different boundary conditions (see appendix B for details). By using (16) one has that the $Z_A(\lambda)$ factors in (20) cancel and the only dependence on $\lambda$ in the integral is in the cutoff propagators. Therefore $\Pi_L^{[1]}$ is equal to the longitudinal part of the photon propagator obtained in the standard perturbation theory, which for large $\Lambda_0$ vanishes as negative powers of $\Lambda_0$ [15]. In perturbation theory renormalizability ensures that $\Lambda_0$ can be sent to infinity and then the longitudinal part of the photon propagator vanishes. In the improved perturbation theory the presence on the Landau pole at $\Lambda_L$ implies that $\Lambda_0$ cannot be removed. Therefore in this case one recovers the transversality of the photon propagator only for momenta much lower than $\Lambda_0$.

The violation of the Ward identity for the vertex is given by the following quantity

$$\Delta_{\alpha\beta}^{[1]}(p,p';0,\Lambda_0) = (p'-p)^\mu\Pi_{\mu\alpha\beta}^{[1]}(p,p';0,\Lambda_0) - e\Pi_{\alpha\beta}^{[1]}(p';0,\Lambda_0) + e\Pi_{\alpha\beta}^{[1]}(p;0,\Lambda_0). \tag{22}$$

Using (18), (19) and (16) one obtains

$$\Delta_{\alpha\beta}^{[1]} = e^3\int_0^{\Lambda_0}\frac{d\lambda}{\lambda}\frac{Z_\psi(\lambda)}{Z_A(\lambda)}[(p'-p)^\mu\hat{I}_{\mu\alpha\beta,irr}(p,p';\lambda) - \hat{I}_{\alpha\beta,irr}(p';\lambda) + \hat{I}_{\alpha\beta,irr}(p;\lambda)], \tag{23}$$

where

$$\hat{I}_{\mu\alpha\beta,irr}(p,p';\lambda) = \hat{I}_{\mu\alpha\beta}(p,p';\lambda) - \hat{I}_{\mu\alpha\beta}(0,0;\lambda)$$

and

$$\hat{I}_{\alpha\beta,irr}(p;\lambda) = \hat{I}_{\alpha\beta}(p;\lambda) - \hat{I}_{\alpha\beta}(0;\lambda) - p^\mu\partial_{\bar{p}^\mu}\hat{I}_{\alpha\beta}(0;\lambda).$$

Notice that setting $Z_A = Z_\psi = \hat{a} = 1$ for any $\lambda$, equation (23) becomes the violation of the usual perturbation theory, which vanishes for $\Lambda_0 \to \infty$ as shown in ref. [15]. The proof is based on the following identity

$$\frac{1}{\not{q}+\not{p}+m}(\not{p}'-\not{p})\frac{1}{\not{q}+\not{p}'+m} = \frac{1}{\not{q}+\not{p}+m} - \frac{1}{\not{q}+\not{p}'+m}$$

and on the fact that in this case the integrand is a total derivative (see (18) and (19)) so that the result of the integration over $\lambda$ is

$$
\begin{aligned}
-ie^3\int_q \frac{K_{0\Lambda_0}(q)}{q^2}\gamma_\rho\Big\{&\frac{K_{0\Lambda_0}(q+p)}{\not q+\not p+m}\gamma^\rho\left[K_{0\Lambda_0}(q+p')-1\right]\\
&+\frac{1}{\not q+m}\gamma_\rho p_\mu\left[\frac{\partial}{\partial\bar p_\mu}K_{0\Lambda_0}(q+\bar p)\right]_{\bar p=0}\Big\}-(p\to p').
\end{aligned}
\tag{24}
$$

For $\Lambda_0\to\infty$ this integral vanishes as negative powers of $\Lambda_0$.

In the improved theory the vanishing of $\Delta^{[1]}_{\alpha\beta}$ can be proved in a similar way. Consider first the contribution in (23) coming from the $g_{\mu\nu}$ part of the photon propagator. One can apply the mean value theorem to extract from the integral the factor $Z_\psi(\bar\lambda)/Z_A(\bar\lambda)$ for some scale $\bar\lambda$, with $0\le\bar\lambda\le\Lambda_0$. This factor multiplies the same integral of the non-improved case. Therefore for large $\Lambda_0$ (but $\Lambda_0 << \Lambda_L$) this contribution vanishes independently of $\bar\lambda$ since $Z_A(\bar\lambda)$ and $Z_\psi(\bar\lambda)$ are at most logarithmically divergent while (24) vanishes as negative powers of $\Lambda_0$. The remaining contribution can be treated in the same way. By applying the mean value theorem one extracts the factor $\frac{Z_\psi}{Z_A}(\hat a-1)$ at some scale $\bar\lambda$, so that also in this case the integrand is a total derivative and the result of the $\lambda$ integration is

$$
\begin{aligned}
-ie^3\int_q \frac{K_{0\Lambda_0}(q)}{q^4}\not q\Big\{&\frac{K_{0\Lambda_0}(q+p)}{\not q+\not p+m}\not q\left[K_{0\Lambda_0}(q+p')-1\right]\\
&+\frac{1}{\not q+m}\not q p_\mu\left[\frac{\partial}{\partial\bar p_\mu}K_{0\Lambda_0}(q+\bar p)\right]_{\bar p=0}\Big\}-(p\to p').
\end{aligned}
\tag{25}
$$

For the argument given above also in this case the result of the $q-$integration vanishes as negative power of $\Lambda_0$. As noted above one cannot remove $\Lambda_0$ due to the Landau pole and therefore also the vertex Ward identity is valid only in a weak sense *i.e.* for momenta much lower than $\Lambda_0$.

# 7 Comparison with the standard improved formulation

In this section we compare our improved perturbation theory with the standard formulation. We show that for non-asymptotically free theories our formulation is very similar to the standard one. To this aim we consider a simple example which trivially generalizes to other interesting cases, *i.e.* the calculation of the improved fish diagram. In the standard resummation approach [12] one passes from a one-loop perturbative quantity to an improved quantity simply replacing the coupling $g$ in the vertices of the Feynman diagrams with the one-loop running coupling $g(q^2)$, where $q$ is the momentum flowing in the loop. This modification means that one has to consider quantities such as

$$\tilde{\mathcal{F}}(Q^2/\mu^2;\Lambda_0^2/\mu^2) = \int_0^{\Lambda_0^2} \frac{dq^2}{2q^2} g^2(q^2)[\tilde{F}(q^2/Q^2) - \tilde{F}(q^2/\mu^2)] \tag{26}$$

with

$$\tilde{F}(q^2/Q^2) = \frac{q^4}{2\pi^2} \int_\Omega \frac{1}{q^2} \frac{1}{(q+Q)^2}, \tag{27}$$

where $\int_\Omega$ indicates the angular integral in four dimension and the $q^4$ factor gives us a dimensionless quantity. On the other hand, in our formulation one has to calculate integrals of the form

$$\mathcal{F}(Q^2/\mu^2;\Lambda_0^2/\mu^2) = \int_0^{\Lambda_0^2} \frac{d\lambda^2}{2\lambda^2} \hat{g}^2(\lambda)[F(\lambda^2/Q^2) - F(\lambda^2/\mu^2)] \tag{28}$$

where $F$ is the function given by (9). We want to show that (26) and (28) are numerically almost the same, *i.e.* $F \simeq \tilde{F}$. In this simple example (27) can be exactly calculated and one gets

$$\tilde{F}(\Lambda^2/Q^2) = \frac{2\Lambda^2}{Q^2+4\Lambda^2} + \frac{8\Lambda^4 \mathrm{ArcTanh}\sqrt{Q^2/(Q^2+4\Lambda^2)}}{\sqrt{Q^2}(Q^2+4\Lambda^2)^{3/2}}.$$

Comparing this result with (10) one finds that $F$ and $\tilde{F}$ have the same asymptotic limits $\Lambda \to 0$ and $\Lambda \to \infty$

$$|\tilde{F} - F| \to \frac{6\Lambda^4}{Q^4} \to 0, \quad \Lambda^2 << Q^2,$$

$$|\tilde{F} - F| \to \frac{1}{15}\frac{Q^2}{\Lambda^2} \to 0, \quad \Lambda^2 >> Q^2$$

and that the relative error $\varepsilon(\Lambda^2/Q^2) = |\tilde{F} - F|/|\tilde{F}|$ is small in any momentum range. The plot of $F$ and $\tilde{F}$ is reported in figure 5.

The fact that the function $F$ is not so different from $\tilde{F}$ can be also seen using in (9) the sharp cutoff function $K_{\Lambda\infty}(q) = \theta(q^2/\Lambda^2 - 1)$ and the approximation $K_{\Lambda\infty}(q+Q) \simeq K_{\Lambda\infty}(q)$. In this case one gets

$$F(\Lambda^2/Q^2) \simeq -8\pi^2 \int_q \Lambda\partial_\Lambda \frac{\theta(q^2/\Lambda^2-1)}{q^2(q+Q)^2} = \tilde{F}(\Lambda^2/Q^2),$$

where we have used the properties $K_{\Lambda\infty}(q)^2 = K_{\Lambda\infty}(q)$ and $\dot{K}_{\Lambda\infty}(q) = -2\Lambda^2\delta(q^2 - \Lambda^2)$. This argument generalizes to one-loop graphs with an arbitrary number of cutoff propagators and various indices (Lorentz, spinor, color, etc) even though these integrals in general cannot be explicitely calculated.

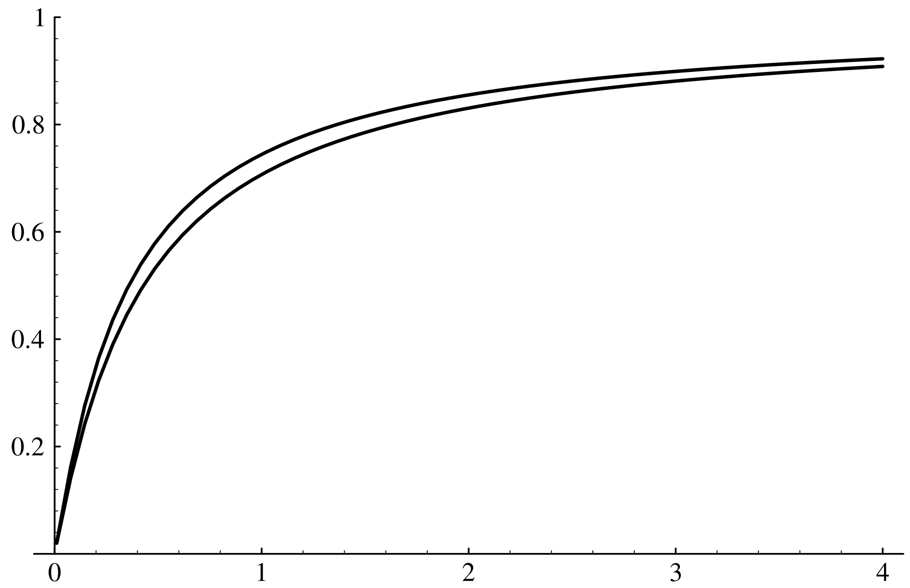


Figure 5: Comparison between the exact result $F(\Lambda^2/Q^2)$ (top curve) and the leading result $\tilde{F}(\Lambda^2/Q^2)$ (bottom curve).

In the approximation $F \simeq \tilde{F}$ our improved theory is completely equivalent to the standard improved theory, with the only difference of replacing the running coupling $g(q^2)$ with the flowing coupling $\hat{g}(\lambda)$. [5] Notice that in the ultraviolet region $g(\sqrt{q^2})$ and $\hat{g}(\lambda)$ have the same functional form, while in the infrared they differ. In particular in the case of asymptotically free theories $\hat{g}(\lambda)$ is regular in all the range of $\lambda$ and the integral (28) is well defined while (26) suffers for the infrared Landau pole.

# 8 Conclusion

We have formulated a systematic improved perturbation theory, based on the Wilson renormalization group. In this approach one solves iteratively the $\Lambda-$RG equations in terms of the Wilsonian flowing coupling $\hat{g}(\lambda)$. In the ultraviolet region this coupling becomes the running coupling constant and our formulation becomes equivalent to the standard improved perturbation theory [14].

In this paper we have considered the $\phi_6^3$ theory and we have shown that the flowing coupling is finite in all the range of $\lambda$ so that our improved perturbation theory is well defined. On the contrary the standard improved perturbation theory is ambiguous due to the infrared Landau pole. We consider this simple model but this result must hold also for the Yang-Mills and QCD theories. A preliminary study of a non-Abelian gauge theory indicates that this is indeed the case [11].

The analysis of a gauge theory requires some care due to the issue of gauge invariance. In this paper we have considered the QED case and we have proved that our improved formulation is consistent with Ward identities, although there are breaking terms because the ultraviolet cutoff cannot be removed. This is due to the presence of the ultraviolet Landau pole in the flowing coupling which reflects the effective character of this theory. Therefore we were able to prove that the first iteration gives a photon two-point function which is transverse only for momenta much lower than the Landau pole. Similarly the Ward identity for the photon-electron vertex is satisfied in this limit. The essential ingredient of the proof is the gauge invariance of the starting point of the iteration. As a conseguence the flowing couplings $\hat{e}(\lambda)$ and $\hat{a}(\lambda)$ are related to the rescaling function $Z_A(\lambda)$. Once the tree level gauge invariance is implemented, the proof follows the same steps of the perturbative case and therefore can be extended to all vertices and, we believe, to all iterations.

The most interesting case is the non-Abelian gauge theory, in which there are several indications that the running coupling constant becomes the effective coupling to use in the Feynman diagrams, but no systematic proof of this assumption is known [13]. The generalization of our method to the non-Abelian case and the proof of the

[5] In QED the analogous of (28) also contains the electron rescaling function, but $Z_\psi(\lambda) \simeq 1$ in the $\lambda << \Lambda_L$ region.

Slavnov-Taylor identities for the one-loop improved effective action is under study.


### Acknowledgments

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# Appendix A

In this appendix we give a method to calculate general cutoff Feynman graphs, specifying an useful form for the cutoff function. The simplest integral one has to compute is

$$F(\Lambda^2/p^2) = -8\pi^2 \int_q \Lambda\partial_\Lambda \left[ \frac{K_{\Lambda\infty}(q)}{q^2} \frac{K_{\Lambda\infty}(q+p)}{(q+p)^2} \right]. \tag{29}$$

This integral can be exactly calculated in polar coordinates using the sharp cutoff function (see [8, 11])

$$K_\Lambda(q) = \theta(q^2/\Lambda^2 - 1). \tag{30}$$

However, with this cutoff function one cannot compute the integrals with more propagators for any $\Lambda$ and for general configurations of the momenta. Moreover the sharp cutoff function (30) is not differentiable and requires some care [8]. To avoid all these problems in this paper we consider the power-law cutoff function

$$K_{\Lambda\infty}(q) = 1 - K_{0\Lambda}(q), \quad K_{0\Lambda}(q) = \frac{\Lambda^4}{(q^2+\Lambda^2)^2}. \tag{31}$$

The cutoff function with both ultraviolet and infrared cutoffs is given by $K_{\Lambda\Lambda_0} \equiv K_{0\Lambda_0} - K_{0\Lambda}$. With this choice of the cutoff function the integral (29) becomes

$$F(\Lambda^2/p^2) = 64\pi^2 \int_q \frac{\Lambda^4}{(q^2+\Lambda^2)^3} \frac{(q+p)^2 + 2\Lambda^2}{((q+p)^2+\Lambda^2)^2}$$

which can be computed using Feynman parameterization formulae. The generalization to integrals with more propagators is straightforward.

Notice that $F(\mathcal{L}^2/p^2) \to 1$ for large $\Lambda >> p$. This behaviour is universal, *i.e.* independent of the precise form of the cutoff function. Indeed setting $K_{\Lambda\infty}(q) \equiv k(x)$, with $x = q^2/\Lambda^2$, one has

$$-8\pi^2 \int_q \Lambda\partial_\Lambda \left( \frac{K_{\Lambda\infty}(q)}{q^2} \right)^2 = -\frac{1}{2} \int_0^\infty \frac{dx}{x} [-2x\partial_x(k(x)^2)] = 1$$

since any cutoff function satisfies to $k(0) = 0$ and $k(\infty) = 1$. For massive fields it is convenient to use the mass-dependent cutoff function

$$K_{\Lambda\infty}(q) = 1 - \frac{\Lambda^4}{(q^2+m^2+\Lambda^2)^2} \tag{32}$$

since in this way the Feynman integrals can be calculated using the Feynman parametrization formulae as in the massless case.

The power-law cutoff function (32) can be extended to $d$ dimensions, $d > 2$, defining

$$K_{\Lambda\infty}(q) = 1 - \left(\frac{\Lambda^2}{q^2+m^2+\Lambda^2}\right)^{[d/2]},$$

where $[d/2]$ indicates the integer part of $d/2$. In particular in $d=6$ one has

$$K_{\Lambda\infty}(q) = (q^2+m^2)\frac{(q^2+m^2)^2+3\Lambda^2(q^2+m^2+\Lambda^2)}{(q^2+m^2+\Lambda^2)^3}, \tag{33}$$

which has been used in section 4 to compute the flowing couplings of the $\phi_6^3$ theory. In particular the functions $F_\phi$ and $F_g$ defined in (13) and computed using (33) are given by

$$F_\phi(\Lambda) = \frac{\Lambda^6(35\Lambda^4+40\Lambda^2m^2+14m^4)}{35(\Lambda^2+m^2)^5}$$

$$F_g(\Lambda) = \frac{\Lambda^6(140\Lambda^8+381\Lambda^6m^2+414\Lambda^4m^4+210\Lambda^2m^6+42m^8)}{140(\Lambda^2+m^2)^7}.$$

# Appendix B

In this appendix we extract the relevant part of the QED cutoff effective action using zero-momentum prescriptions (for the case of on-shell renormalization prescriptions see for instance [15]). This relevant functional is given by

$$\begin{aligned}\Pi_{rel}(\phi;\Lambda,\Lambda_0) &= \int_k -\tfrac{1}{2}A_\mu(-k)(k^2g^{\mu\nu}Z_A + g^{\mu\nu}\sigma_2 + k^\mu k^\nu\sigma_\xi)A_\nu(k) \\ &\quad + \int_p \bar\psi(p)(\not{p}Z_\psi - \sigma_m)\psi(p) + \int_x \sigma_e\bar\psi\not{A}\psi + \tfrac{1}{8}\sigma_4(A_\mu A^\mu)^2.\end{aligned} \tag{34}$$

The relevant couplings are defined by

$$Z_A = -\,\partial_{k^2}\frac{1}{3}t^{\mu\nu}\Pi_{\mu\nu}\Big|_{k=0}, \quad \sigma_2 = -\frac{1}{3}t^{\mu\nu}\Pi_{\mu\nu}\Big|_{k=0}, \quad \sigma_\xi = \frac{1}{3}\partial_{k^2}o^{\mu\nu}\Pi_{\mu\nu}\Big|_{k=0},$$

$$Z_\psi = \partial_{p^\mu}\frac{1}{16}\gamma^\mu_{\beta\alpha}\Pi_{\alpha\beta}\Big|_{p=0}, \quad \sigma_m = -\,\frac{1}{4}\Pi_{\alpha\alpha}\Big|_{p=0}, \quad \sigma_e = \frac{1}{16}\gamma^\mu_{\alpha\beta}\Pi_{\mu\beta\alpha}\Big|_{p_i=0},$$

$$\sigma_4 = \frac{1}{72}(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}+g^{\mu\rho}g^{\sigma\nu})\Pi_{\mu\nu\rho\sigma}\Big|_{p_i=0}$$

where $\Pi_{\mu\nu}$ and $\Pi_{\alpha\beta}$ are the photon and electron two-point functions, $\Pi_{\mu\alpha\beta}$ and $\Pi_{\mu\nu\rho\sigma}$ are the photon-electron and the four-photon vertices and

$$t_{\mu\nu} = g_{\mu\nu} - k_\mu k_\nu/k^2, \qquad s_{\mu\nu} = g_{\mu\nu} - 4k_\mu k_\nu/k^2.$$

At the tree level one has

$$\Pi^{(0)} = \int_k -\frac{1}{2} A_\mu (k^2 g_{\mu\nu} + k^\mu k^\nu (\frac{1}{a} - 1)) A_\nu + \int_p \bar\psi (\not p - m)\psi + \int_x e\bar\psi \not A \psi$$

*i.e.*

$$\sigma_A^{(0)} = 1, \quad \sigma_\xi^{(0)} = \frac{1}{a} - 1, \quad \sigma_2^{(0)} = 0,$$

$$\sigma_\psi^{(0)} = 1, \quad \sigma_m^{(0)} = m, \quad \sigma_e^{(0)} = e, \quad \sigma_4^{(0)} = 0.$$

The irrelevant parts of the photon and electron two-point functions are given by

$$\Pi_{\mu\nu,irr}(k) = \Pi_{\mu\nu}(k) - \left[\frac{1}{3} t^{\rho\sigma} \Pi_{\rho\sigma}(\bar k)\right]_{\bar k=0} g_{\mu\nu} - \left[\frac{1}{3} \partial_{\bar k^2} (t^{\rho\sigma} \Pi_{\rho\sigma}(\bar k))\right]_{\bar k=0} k^2 g_{\mu\nu}$$

$$+ \left[\partial_{\bar k^2} s^{\rho\sigma} \Pi_{\rho\sigma}(\bar k)\right]_{\bar k=0} k_\mu k_\nu$$

and

$$\Pi_{\alpha\beta,irr}(p) = \Pi_{\alpha\beta}(p) - \left.\Pi_{\alpha\beta}(\bar p)\right|_{\bar p=0} - p_\mu \left.\frac{\partial}{\partial \bar p_\mu} \Pi_{\alpha\beta}(\bar p)\right|_{\bar p=0}.$$

The irrelevant parts of the photon-electron vertex and the four-photon vertex are given by

$$\Pi_{\mu\alpha\beta,irr}(p,p') = \Pi_{\mu\alpha\beta}(p,p') - \Pi_{\mu\alpha\beta}(0,0)$$

and

$$\Pi_{\mu\nu\rho\sigma,irr}(p,q,r) = \Pi_{\mu\nu\rho\sigma}(p,q,r) - \Pi_{\mu\nu\rho\sigma}(0,0,0).$$

The same decomposition into relevant and irrelevant parts holds for the vertices of the functional $I$ and also in the improved case, *i.e.* for the functional $\hat\Pi$ and $\hat I$.

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