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# Gauge Consistent Wilson Renormalization Group II: Non-Abelian case

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## Abstract

We give a Wilsonian formulation of non-Abelian gauge theories explicitly consistent with axial gauge Ward identities. The issues of unitarity and dependence on the quantization direction are carefully investigated. A Wilsonian computation of the one-loop QCD beta function is performed.

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# 1 Introduction

In a recent paper [1] we succeeded to give a continuous Wilsonian formulation of Abelian gauge theories explicitly consistent with gauge-invariance. The basic idea was of introducing the Wilsonian infrared cutoff $\Lambda$, which separates the soft from the hard modes, as a mass term for the propagating fields. In this way the Wilson's Renormalization Group Equation (also called Exact Renormalization Group Equation, ERGE) becomes consistent with the Ward-Takahashi identities to all scales. The price to pay is the need for an explicit regularization and renormalization of the evolution equation, since with the mass cutoff the ultraviolet momenta are not sufficiently suppressed. However this is a little price to pay since, at least in perturbation theory, there exist gauge-invariant techniques to manage the ultraviolet divergences (dimensional regularization with minimal subtraction, higher derivatives approaches, etc.).

In this paper we introduce a suitable generalization working in the non-Abelian case. The key idea is to use an algebraic non-covariant gauge. This is quite natural since in that kind of gauges we can recover as much as possible the properties of the Abelian Ward identities and we can implement the method developed in [1]. There are various gauges belonging to the general class of algebraic non-covariant gauges [2, 3]: the axial gauge, the planar gauge and the light-cone gauge. Of these the light-cone gauge is the most famous and the most solid from a theoretical point of view. Nevertheless, in this paper, we will restrict to the analysis of the pure axial gauge, which is technically the simplest to manage. Actually, it is known that this choice has non-trivial infrared problems related to a proper definition of the spurious singularities: in particular the Wilson loop consistency tests fails with the Cauchy Principal Value (CPV) prescription [4]. In that paper we will use a regularization of spurious divergences different from the CPV prescription, nevertheless we still expect a very delicate infrared limit. The study of that problem, which eventually could require the switching to a more established gauge, as the light-cone one, will be the subject of a forthcoming paper [5]. Herein, for sake of space, we will restrict to the analysis of Ward identities and ultraviolet properties. A great amount of what we will discuss for the pure-axial gauge case can be extended to the other non-covariant gauges.

Non-covariant gauges have been implemented in the ERGE formalism in [6] (for the axial gauge) and [7] (for the light-cone gauge). However, in these references a generic Wilsonian cutoff function has been employed. As a consequence the standard Ward identities are broken. In other words, even if we start from a gauge-invariant ultraviolet action, the evolution generates non-gauge-invariant structures (spurious couplings). Actually these couplings are constrained by some modified Ward identities (mWI's) [6, 8, 9] which becomes the usual ones only at the physical scale $\Lambda = 0$. Although these mWI's are perfectly understood at the formal level, they are quite difficult to study in practice. In particular, in order to perform a consistent computation, we should provide an initial (ultraviolet) action consistent with the mWI's. From a perturbative point of view, this is not a problem since it is well known [9, 10] that it is possible to choose the boundary conditions on the spurious couplings in terms of the physical couplings in such a way that the mWI's hold to all scales. This property is a consequence of the Quantum Action Principle [11] and holds to any order in perturbation theory, provided that the theory is anomaly free. However, this approach involves a highly non-trivial fine-tuning procedure which is extremely cumbersome beyond the one-loop level[2]. Moreover at the non-perturbative level no non-trivial truncations of the effective action consistent with the mWI's are known. The same problem also holds in covariant gauges and it is even worse in this case, since the modified Slavnov-Taylor identities (mST's) [8, 9] are more cumbersome.

The solution of this technical difficulty motivates our work. In fact, we will prove that, by using a specific choice of the infrared cutoff, the *usual* form of the Ward identities can be maintained to *all* scales. In other words, in our formulation the evolution equation is gauge-invariant and the renormalization group flow preserves the gauge-symmetry. Two different proofs and a practical check of this issue will be provided in this paper.

Moreover, the question of unitarity will be examined in detail.
Clearly, in a generic Wilsonian procedures unitarity is spoiled due to the breaking of gauge symmetry. In our formalism instead unitarity can be maintained for any $\Lambda$, even in the non-Abelian case. This fact will be proved by explicitly showing the compatibility of the propagator with the Landau-Cutkosky rules. However, as it should be expected, the theory is *not* physically consistent at $\Lambda \neq 0$, since there is an unavoidable dependence of "physical" quantities on the quantization direction $n_\mu$. In other words there is an unphysical breaking of Lorentz-covariance. In order to control this problem (which is generic for any Wilsonian formulation of the axial

[2] Even at one-loop the fine-tuning is not so simple; see [12] for an analysis in the covariant Feynman gauge.

gauge) we introduce a generalized BRST symmetry holding also at $\Lambda \neq 0$. In this way a simple analysis of the $n_\mu-$dependence is possible in terms of generalized Slavnov-Taylor identities. In particular we show formally (i.e. modulo problems with infrared divergences, which will be addressed in [5]) that when the infrared cutoff is removed the Lorentz-covariance of the physical theory is recovered. We stress that in general the control of the unphysical $n_\mu-$dependence cannot be neglected since in numerical analysis the $\Lambda \to 0$ limit is never reached.

Finally, we check the practical reliability of the formulation with some perturbative computation: in particular we recover, with a Wilsonian non-standard computation, the usual universal value of the one-loop QCD beta function.

The plan of the paper is the following: section 2 contains our notations and conventions; section 3 is a simple introduction to the axial gauge in presence of an explicit mass term; in section 4 we briefly discuss the Wilsonian (perturbative) renormalization of the theory. In section 5 we check the Ward identities at one-loop with elementary methods: in particular we study the transversality property of the gluon propagator. In section 6 and 7 we give the general proof of gauge invariance by following two different approaches. Section 8 faces the problems of unitarity and gauge-dependence at the formal level, by using BRST techniques. In section 9, as a consistency check of the formalism, we compute the one-loop QCD beta function with Wilsonian techniques. Section 10 contains our conclusions. Three appendices concerning technical points close the paper. Appendix A contains some comments about the renormalizability property in the axial gauge framework from a Wilsonian point of view. In appendix B the rigorous deduction of the modified Ward identities is reviewed and the relation with the Quantum Action Principle and the fine tuning problem is clarified. Appendix C contains some useful trick to make perturbative computations in our formalism in an efficient way.

# 2 Notations and conventions

For future reference and for commodity of the reader, we begin by collecting our conventions and some useful formula.
We work in QCD with $N_C$ colors and $N_f$ flavors. The gauge fields are denoted by $A_\mu^a(x)$; the matter fields (quarks and antiquarks) with $\psi^i(x)$ and $\bar{\psi}_i(x)$, where $i$ is the flavor index; spinor and color indices are understood. The

generators of the group in the fundamental representation are denoted by $T_a$ and are taken hermitian $T_a = T_a^+$; the generators in the adjoint representation are denoted by $\tau_a$ and corresponds to the structure constants $(\tau_a)_{bc} = f_{abc}$. We use the condensed notations $X \cdot Y = \delta_{ab} X^a Y^b$ and $(X \times Y)^a = f^a_{bc} X^b Y^c$ for the inner and the outer product in the adjoint representation. The covariant derivative $D_\mu$ acts as $D_\mu X = \partial_\mu X + g A_\mu \times X$ and $D_\mu \psi = \partial_\mu \psi - i g A^a_\mu T_a \psi$; $g$ is the coupling constant. The gauge transformations are generated by the functional operator

$$\mathcal{W}_f = - \int d^4x f(x) \cdot \left[ D_\mu \frac{\delta}{\delta A_\mu} + i g \bar{\psi} T \frac{\delta}{\delta \bar{\psi}} - i g T \psi \frac{\delta}{\delta \psi} \right] \tag{1}$$

and on the fundamental fields we have the relations

$$\mathcal{W}_f A_\mu = D_\mu f, \quad \mathcal{W}_f \psi = i g f \cdot T \psi, \quad \mathcal{W}_f \bar{\psi} = - i g f \cdot \bar{\psi} T. \tag{2}$$

The field strength tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu$ and the gauge-invariant classical action is

$$S_{CL}(A, \psi, \bar{\psi}) = S_{CL,gauge}(A) + S_{CL,matter}(A, \psi, \bar{\psi}) \tag{3}$$

with

$$S_{CL,gauge}(A) = - \frac{1}{4} \int d^4x \; F_{\mu\nu} \cdot F^{\mu\nu}, \tag{4}$$

$$S_{CL,matter}(A, \psi, \bar{\psi}) = \int d^4x \; \bar{\psi}_i (i \not{D} - m^i_j) \psi^j. \tag{5}$$

Here $m^i_j$ denotes the (diagonal) quark mass matrix. We will use the tensors

$$t_{\mu\nu}(p) = g_{\mu\nu} - p_\mu p_\nu / p^2, \quad \ell_{\mu\nu} = p_\mu p_\nu / p^2, \tag{6}$$

$$V_{\mu\nu\rho}(p, q, r) = (q - r)_\mu g_{\nu\rho} + (p - q)_\rho g_{\mu\nu} + (r - p)_\nu g_{\rho\mu}, \tag{7}$$

$$\begin{aligned} t^{abcd}_{\mu\nu\rho\sigma} = (f^{acbd} - f^{adcb}) g_{\mu\nu} g_{\rho\sigma} + (f^{abcd} - f^{adbc}) g_{\mu\rho} g_{\nu\sigma} + \\ (f^{acdb} - f^{abcd}) g_{\mu\sigma} g_{\nu\rho}, \quad f^{abcd} \equiv f^{ab}_e f^{ecd}. \end{aligned} \tag{8}$$

The following classical Ward identities holds, as a consequence of gauge-invariance:

$$p^\mu t_{\mu\nu}(p) = 0 \tag{9}$$

$$p_1^\mu V_{\mu\nu\rho}(p_1, p_2, p_3) = p_3^2 t_{\nu\rho}(p_3) - p_2^2 t_{\nu\rho}(p_2) \tag{10}$$

$$\begin{aligned} p_1^\alpha t^{abcd}_{\alpha\beta\gamma\delta} = &- f^{abcd} V_{\beta\gamma\delta}(p_1 + p_2, p_3, p_4) - f^{acdb} V_{\gamma\delta\beta}(p_1 + p_3, p_4, p_2) \\ &- f^{adbc} V_{\delta\beta\gamma}(p_1 + p_4, p_2, p_3). \end{aligned} \tag{11}$$

In (10) and (11) the vanishing of the sum $\sum_i p_i = 0$ is understood. These identities will be used in the study of the one-loop quantum Ward identities. In the text we will use the deWitt condensed notation, by denoting with

$$\Phi^A = (A^a_\mu(p), \psi_i(p), \bar{\psi}^i(p))$$

all the fields of theory (when required in BRST considerations, we will also collect in $\Phi^A$ the ghosts and the auxiliary fields). The 1PI Green functions are generally denoted by

$$\Gamma_{A_1 \dots A_n} = \left. \frac{\delta\Gamma}{\delta\Phi^{A_1} \dots \delta\Phi^{A_n}} \right|_{\Phi=0}. \tag{12}$$

For instance the two-point function $\Gamma_{AB}$ both corresponds to the gluon propagator $(2\pi)^4\delta(p+q)\Gamma_{\mu\nu}(p)$ and to the quark propagator $(2\pi)^4\delta(p+q)\Gamma_{ij}(p)$. Sums and integrals over repeated indices are understood and the supertrace notation is used, $\mathrm{STr}X = (-)^A X^A_A$, where $(-)^A$ has the value $+1$ for bosonic fields and $-1$ for fermionic fields.

Some useful abbreviations on integrals are

$$\int_x = \int d^4x, \quad \int_q = \int \frac{d^4q}{(2\pi)^4}, \quad \int_{q_3} = \int \frac{dq_3}{2\pi}, \quad \int_{\bar{q}} = \int \frac{d^3\bar{q}}{(2\pi)^3}, \tag{13}$$

with $\bar{q} = (q^0, q^1, q^2)$. For Euclidean vectors we shall use the notations

$$p_E = (ip_0, \vec{p}), \quad q_E = (iq_0, \vec{q}), \quad p_E q_E \equiv \delta_{\mu\nu}\, p_E^\mu\, q_E^\nu = -g_{\mu\nu} p^\mu q^\nu = -pq. \tag{14}$$

If not otherwise specified, all the quantities should be intended in the Minkowski space.

Equipped with these conventions, we can begin our analysis with some simple consideration on the gauge-fixing procedure from a Wilsonian perspective.

## 3 Remarks on the gauge-fixing procedure

In order to define a perturbative quantum field theory from a classical gauge invariant field theory one is forced to break gauge invariance. This breaking is required to define an invertible propagator for the gauge field and is technically done by adding to the classical (free) action

$$S_{CL}(A) = \int_x -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\int_p \frac{1}{2} A_\mu(-p) t^{\mu\nu}(p) p^2 A_\nu(p) \tag{15}$$

a gauge fixing term. Typically, this term is quadratic in the gauge fields i.e. only affects the propagators and not the vertices of the theory. Therefore the gauge fixed action has the general form

$$\Gamma^{(0)}_{gauge}(A) = S_{CL}(A) + S_{GF}(A), \quad S_{GF}(A) = \frac{1}{2} \int_p A_\mu(p) Q^{\mu\nu}(p) A_\nu(p) \tag{16}$$

where $Q_{\mu\nu}(p) = Q_T(p) t_{\mu\nu}(p) + Q_L(p) \ell_{\mu\nu}(p)$ is a suitable symmetric tensor, to be taken such as the propagator of the gauge fields

$$-D_{\mu\nu}(p) = \big(p^2 t_{\mu\nu}(p) - Q_{\mu\nu}(p)\big)^{-1} = \frac{t_{\mu\nu}(p)}{p^2 - Q_T(p) + i\varepsilon} + \frac{\ell_{\mu\nu}(p)}{-Q_L(p) + i\varepsilon} \tag{17}$$

satisfies regularity properties both in the ultraviolet and in the infrared. The more general gauge fixing quadratic in the fields consistent with Lorentz symmetry and the power counting criterium is given by the formula [1]

$$Q_{\Lambda,\mu\nu}(p) = -\frac{1}{\xi} p_\mu p_\nu + \Lambda^2 g_{\mu\nu}, \tag{18}$$

where $\xi$ is a generic[3] dimensionless parameter which is expected do not affect the physics whereas the scale $\Lambda$ is interpreted as a Wilsonian cutoff distinguishing between soft ($p_E^2 << \Lambda^2$) and hard ($p_E^2 >> \Lambda^2$) modes. With this choice the Euclidean propagator becomes regular both in the infrared

$$D_{\Lambda,\mu\nu}(p_E) \sim 1/\Lambda^2, \quad p_E^2 << \Lambda^2, \tag{19}$$

and in the ultraviolet

$$D_{\Lambda,\mu\nu}(p_E) \sim 1/p_E^2, \quad p_E^2 >> \Lambda^2, \tag{20}$$

[3] The only requirement is $\xi < \infty$ in order the propagator satisfies the power counting criterium (20). In the limit $\xi \to \infty$ one recovers the Proca theory [17].

and in particular satisfies the power counting criterium which is essential in the renormalizability proof.

In the Abelian case this simple procedure is enough to construct a consistent quantum field theory in the framework of the ERGE [1]. The reason of this success is clear: even if gauge invariance is formally broken, nevertheless the theory still preserve all the good properties of the gauge symmetry since the tree-level Ward-Takahashi identities are only linearly broken,

$$\mathcal{W}_f \Gamma^{(0)}(A,\xi;\Lambda) = \int_x f(x) \left( \frac{\Box}{\xi} + \Lambda^2 \right) \partial_\mu A^\mu. \tag{21}$$

This exceptional property guarantees that identity (21) can be lifted to the quantum level, provided that we use a consistent renormalization procedure [4]. Essentially, in [1] we have proved that the Wilson's Renormalization Group Equation provides such a consistent procedure. Therefore perturbative corrections to the tree level action are gauge invariant to all orders,

$$\mathcal{W}_f \Gamma^{(\ell)}(A,\xi;\Lambda) = 0, \quad \forall \ell \geq 1. \tag{22}$$

As it is well known [17] this fact assures the unitarity of the Abelian theory to all scales $\Lambda$. However, in the non-Abelian case, this procedure does not work, since the breaking term is quadratic in the fields:

$$\mathcal{W}_f \mathcal{S}_{GF} = \frac{1}{2} \int_x A_\mu \cdot Q_\Lambda^{\mu\nu} D_\nu f + D_\mu f \cdot Q_\Lambda^{\mu\nu} A_\nu. \tag{23}$$

For this reason the gauge symmetry cannot be lifted to the quantum level and the quantum corrections are *not* gauge-invariant. The standard way to solve this problem is of introducing the ghost fields $C^a(x)$, $\bar{C}^a(x)$ and the Nakanishi-Lautrup auxiliary fields $B^a(x)$, substituting the gauge symmetry with the BRST symmetry [18]

$$sA_\mu = D_\mu C, \quad sC = -\frac{1}{2} g C \times C, \quad s\bar{C} = B, \quad sB = 0. \tag{24}$$

In this way we can take

$$S_{BRST} = S_{CL} + s\left( \bar{C} \cdot \partial_\mu A^\mu + \frac{1}{2}\xi \bar{C} \cdot B \right) \tag{25}$$

[4] Needless to say, such a procedure does not exist in many interesting cases, in particular in chiral models.

as a suitable tree level action from which a renormalization program consistent with the symmetry can start. However, there is a price to pay: the BRST symmetry is *non-linear* in quantum fields and it can be lift to the quantum level only in the form of Slavnov-Taylor identities which renormalization is technically more complicate. Moreover a detailed study of graphs involving ghost particles is required in doing perturbative computations. Actually, as it is well known, one can avoid these technical complications if one relaxes the requirement of covariance and works in a particular class of non-covariant gauges (actually this was the original motivation in studying axial gauges, also called ghost-free gauges or physical gauges [23]).

Here we are interested in the pure axial gauge which has been extensively used both in theoretical and in phenomenological literature (for a review see [2, 3]). However, since we have in mind a Wilsonian interpretation, our analysis will be non-standard, based on a massive version of the usual axial gauge-fixing. For definiteness we will take as gauge fixing term

$$S_{GF}(A,\xi_2;\Lambda) = \int_x \frac{1}{2\xi_2 n^2} n^\mu A_\mu \cdot n^\nu A_\nu + \frac{1}{2}\Lambda^2 A_\mu \cdot A^\mu, \tag{26}$$

where $n^\mu$ is a space-like vector, $n^2 < 0$. In particular the Arnowitt-Ficker choice $n^\mu = (0,0,0,1)$ will be considered in the beta function computation in section 9. In terms of the cutoff function $Q_{\Lambda,\mu\nu}$ the axial gauge fixing corresponds to the choice

$$Q_{\Lambda,\mu\nu} = g_{\mu\nu}\Lambda^2 + \frac{1}{\xi_2}\frac{n_\mu n_\nu}{n^2}. \tag{27}$$

Notice that in (26) we have introduced a non-renormalizable auxiliary parameter $\xi_2$ of mass dimension $-2$. However in this paper we will only consider the limit $\xi_2 \to 0$ (pure axial gauge[5]). In this case the propagator has the form

$$\begin{aligned}-D_{\Lambda,\mu\nu}(p) \overset{\xi_2\to 0}{=} & \frac{1}{p^2-\Lambda^2+i\varepsilon} g_{\mu\nu} - \frac{p\cdot n(n_\mu p_\nu + n_\nu p_\mu)}{(p^2-\Lambda^2+i\varepsilon)((p\cdot n)^2 - n^2\Lambda^2)} + \\ & \frac{p_\mu p_\nu \, n^2}{(p^2-\Lambda^2+i\varepsilon)((p\cdot n)^2 - n^2\Lambda^2)} + \frac{\Lambda^2 n_\mu n_\nu}{(p^2-\Lambda^2+i\varepsilon)((p\cdot n)^2 - n^2\Lambda^2)}.\end{aligned} \tag{28}$$

[5]One can easily convince himself that $\xi_2 = 0$ is a fixed point of the ERGE, and actually this property holds for any choice of the cutoff function [6]. Therefore $\xi_2 = 0$ is preserved by the evolution.

The limit $\xi_2 \to 0$ has particular properties since $D_{\Lambda,\mu\nu}(p)$ becomes transverse

$$n^\mu D_{\Lambda,\mu\nu}(p) \stackrel{\xi_2 \to 0}{=} 0, \quad \forall \, \Lambda \tag{29}$$

and therefore is not invertible (for any $\Lambda$). This is the same phenomenon which happens in the covariant Landau gauge. The naïve zero mass limit of (28) gives the usual form of the axial gauge propagator [23]: however this limit is quite delicate since there are spurious singularities at $n^\mu p_\mu = 0$. Such infrared singularities are automatically regularized at $\Lambda \neq 0$. This is a typical feature of the Wilsonian approach, also noticed in [6]. These authors in fact computed the general form of the propagator with a generic cutoff $R_{\Lambda,\mu\nu}(p)$ of which (29) is a particular case. However, this particular case is exceptionally important since in this case the gauge-fixing term breaks gauge invariance only in a linear way [6]

$$\mathcal{W}_f S_{GF} = \int_x \frac{1}{\xi_2 n^2} n_\mu A^\mu \cdot n_\nu \partial^\nu f + \Lambda^2 A^\mu \cdot \partial^\nu f = \int_x A^\mu \cdot Q_{\Lambda,\mu\nu} \partial^\nu f. \tag{30}$$

Therefore in this case the situation is QED-like and the classical gauge symmetry can be lifted to the quantum level in a consistent way.

The same result holds when the matter coupling is considered, provided that one introduces the Wilsonian infrared cutoff as a mass term for the matter fields, as has been done in [1], by taking

$$\Gamma^{(0)}_{matter}(A, \bar\psi, \psi; \Lambda) = \int_x \bar\psi_i i \not{D} \psi^i - \bar\psi_i (m^i_j + i\Lambda\gamma_5 \delta^i_j)\psi^j. \tag{31}$$

In the following we will also use the general notation[7]

$$\Gamma^{(0)}(\Phi; \Lambda) = S_{CL}(\Phi) + \frac{1}{2}\tilde\Phi Q_\Lambda \Phi, \tag{32}$$

with

$$\frac{1}{2}\tilde\Phi Q_\Lambda \Phi \equiv \frac{1}{2}\int_p A_\mu(-p) Q^{\mu\nu}_\Lambda(p) A_\nu(p) - \int_p \bar\psi_j(p) i\Lambda\gamma_5 \psi^j(p), \tag{33}$$

[6]In the deduction of (30) the identities $n_\mu A^\mu \cdot n_\nu A^\nu \times f = n_\mu n_\nu A^\mu \times A^\nu \cdot f = 0$ and $A_\mu \times A^\mu \cdot f = 0$. should be used.

[7]A more explicit notation is $\frac{1}{2}\tilde\Phi Q_\Lambda \Phi = \frac{1}{2}(-)^A \Phi^A Q_{\Lambda,AB} \Phi^B = \frac{1}{2} Q_{\Lambda,AB} \Phi^B \Phi^A$. The tilde denotes the transposition of the generalized indices with suitable minus signs for fermionic entries.

since a geometric interpretation becomes more clear: the (graded) symmetric invertible tensor $Q_{\Lambda,AB}$ can be seen as a metric in the field space. Thus, there is an explicit invariance of the gauge-fixing term under isometries, i.e. linear transformation $\Phi^A \to U^A_B \Phi^B$ such as $\tilde{U}^A_C Q_{\Lambda,AB} U^B_D = Q_{\Lambda,CD}$. This interpretation will be useful to give a clear proof of gauge-invariance in section 6.

# 4 Wilsonian perturbative renormalization

Having defined the tree level theory, i.e. the functional $\Gamma^{(0)}(\Phi;\Lambda)$, one can compute the quantum correction $\Gamma^{(\ell)}(\Phi;\Lambda)$ to any order in perturbation theory by introducing a consistent renormalization procedure. In this paper we shall adopt the renormalization procedure based on the ERGE introduced in [1] (see also [14, 15, 16]). By referring to [1] for a complete discussion, here we simply write down the explicit form of the equation for the proper vertices, which reads

$$\dot{\Pi}_{A_1\ldots A_n} = \hbar I_{A_1\ldots A_n} = -\hbar \left[ \frac{i}{2} (-)^A (\Gamma_2^{-1} \dot{Q}_\Lambda \Gamma_2^{-1})^{BA} \bar{\Gamma}_{AA_1\ldots A_n B} \right]_{reg} . \tag{34}$$

We remind the notation:

- $\Pi(\Phi;\Lambda)$ is the gauge-invariant effective action

$$\Pi(\Phi;\Lambda) \equiv \Gamma(\Phi;\Lambda) - \frac{1}{2}\tilde{\Phi} Q_\Lambda \Phi \tag{35}$$

  which satisfies the non-Abelian Ward-Takahashi identities

$$\mathcal{W}_f \Pi = \int_x D_\mu f \cdot \frac{\delta \Pi}{\delta A_\mu} = - \int_x f \cdot \left( \partial_\mu \frac{\delta \Pi}{\delta A_\mu} + g A_\mu \times \frac{\delta \Pi}{\delta A_\mu} \right) = 0 \tag{36}$$

  as we will prove.

- The auxiliary vertices $\bar{\Gamma}_{AA_1\ldots A_n B}$ are recursively obtained from the usual vertices via the formula [15]

$$\bar{\Gamma}_{AA_1\ldots A_n B} = \Gamma_{AA_1\ldots A_n B} - \sum_{k=1}^{n-1} \Gamma_{AA_1\ldots A_k C} (\Gamma_2^{-1})^{CD} \bar{\Gamma}_{DA_{k+1}\ldots A_n B} . \tag{37}$$

  as shown in figure 1.

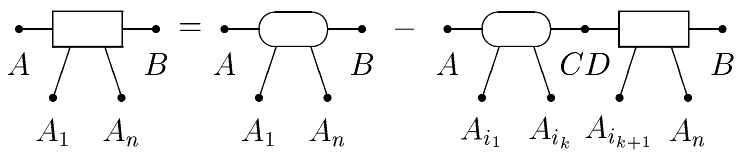


Figure 1: Recursive expansion of the $\bar{\Gamma}_{AA_1\ldots A_nB}$ vertices, denoted by the boxes. The black dots denote the full propagators and the ovals the full vertices. A sum over inequivalent permutations of external lines is understood.

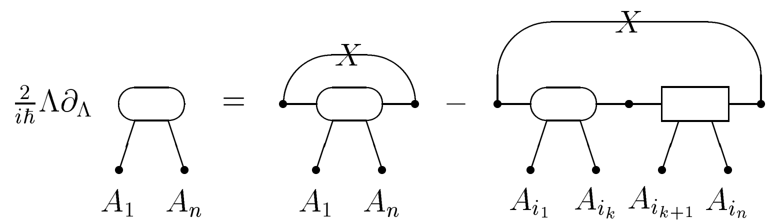


Figure 2: Diagrammatic version of the exact evolution equation. Here $X = \dot{Q}_\Lambda$.

- The subscript $reg$ in (34) denotes some ultraviolet regularization needed to properly define the loop integrals in the evolution equation. In fact, as we explained in detail in [1], in four dimensions the two-point functions $\dot{\Pi}_{AB}$ are logarithmically divergent and must be regularized and renormalized. In the following we shall consider both the effects of a gauge-consistent and of a gauge-inconsistent regularization.

A graphical representation of the evolution equation is reported in figure 2.

Notice that in (34) we have explicited the factor $\hbar$ in order to clarify how the ERGE acts as an iterative renormalization procedure. In fact, as fully reviewed in [1] (see also [13] and [15]) once boundary conditions on relevant couplings (i.e. power counting renormalizable couplings in a more common terminology) are fixed, one can solve iteratively the ERGE to all orders in perturbation theory by recovering in practice the BPHZ subtracted proper vertices. In the case at hand, since Lorentz covariance is broken, the analysis of relevant couplings differs from the covariant one and the more general gauge-invariant relevant functional has the form

$$\Pi_{rel}(\Phi;\Lambda) = \int_x -\frac{1}{4}Z_A(\Lambda)F_{\mu\nu}\cdot F^{\mu\nu} - \frac{1}{2}h(\Lambda)\frac{n^\mu n_\sigma}{n^2}F_{\mu\rho}\cdot F^{\rho\sigma}. \tag{38}$$

The relevant parameters $Z_A(\Lambda)$ and $h(\Lambda)$ can be obtained from the two-point function which has the form[8]

$$\Pi_{\mu\nu,rel}(p,n;\Lambda) = Z_A(\Lambda)p^2 t_{\mu\nu}(p) + h(\Lambda)p^2 T_{\mu\nu}(p,n) \tag{39}$$

with

$$T_{\mu\nu}(p,n) = \frac{(np)^2}{n^2p^2}g_{\mu\nu} - \frac{np}{n^2p^2}(pn+np)_{\mu\nu} + \frac{n_\mu n_\nu}{n^2}. \tag{40}$$

Here we adopt zero-momentum prescriptions and we fix the relevant couplings at some non-zero infrared renormalization scale $\Lambda_R$:

$$Z_A(\Lambda_R) = 1, \quad h(\Lambda_R) = 0. \tag{41}$$

The generation of the new relevant coupling $h(\Lambda)$ is due to the Lorentz-covariance breaking. However, this coupling is *not* an independent coupling and its evolution is constrained from a non-linear functional identity describing the dependence on the quantization direction $n_\mu$. Moreover at one-loop $h(\Lambda)$ is zero. All these issues will be examined in detail in section 7. Here we are more interested on the function $Z_A(\Lambda)$ which is physically very important since it is related to the coupling constant[9] $g(\Lambda)$ via the QED-like relation

$$g(\Lambda) = \frac{g(\Lambda_R)}{Z_A^{1/2}(\Lambda)}. \tag{42}$$

This relation is imposed from the gauge symmetry and is *exact* in our formalism whatever vertex is used to define $g(\Lambda)$ (at zero momentum the $\bar\psi A\psi$ vertex, the $AAA$ vertex and the $AAAA$ vertex give the same coupling, since gauge-invariance is preserved).

[8]Notice that both $t_{\mu\nu}(p)$ and $T_{\mu\nu}(p,n)$ are transverse as a consequence of the Ward identity $p^\mu\Pi_{\mu\nu}(p,n;\Lambda) = 0$.

[9]Clearly the coupling $g(\Lambda_R)$ can be related to the $g(\mu)$ coupling of some momentum dependent prescription, $g(\mu) = f(g(\Lambda_R), \mu/\Lambda_R)$ and the explicit form of $f$ can be computed in perturbation theory.

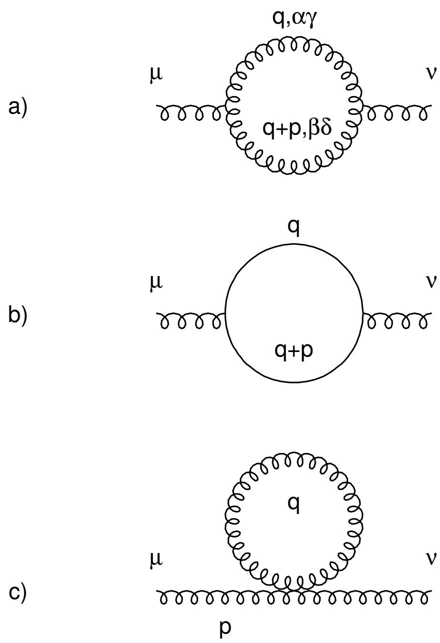


Figure 3: One-loop Feynman diagrams contributing to the evolution of the gluon propagator. Color indices are understood.

The important point we want to stress is the following: since our formalism is Ward-identities-consistent, no spurious couplings are generated and *no fine tuning* is required for them.

Having fixed the boundary conditions, in principle one can solve the evolution equation to all orders in perturbation theory in terms of integrals over the scale $\Lambda$ since the renormalizability property guarantees that they are all well defined. This point is fully treated in [1] and can be lifted to the present case without problems (we refer to appendix A for some additional comment). Instead, it seems worthwhile of explaining how the Renormalization Group Equation works in practice, by explicitly solving it at one-loop and by checking the consistence with the Ward-Takahashi identities. In particular we will study the evolution equation for the one-loop gluon propagator, which has the general form

$$\dot{\Pi}^{(1)}_{\mu\nu,ab}(p,n;\Lambda) = \Lambda\partial_\Lambda F^{reg}_{\mu\nu,ab}(p,n;\Lambda) \tag{43}$$

where the function $F^{reg}_{\mu\nu,ab}(p,n;\Lambda)$ denotes the sum of the usual regularized one-loop Feynman diagrams (figure 3). For instance the contribution of figure 3a is given by

$$F^{(a)reg}_{\mu\nu,ab}(p,n;\Lambda) = \frac{ig^2}{2}\delta_{ab}\int_q [V^{\mu\alpha\beta}(p,q,-q-p) \cdot D_{\Lambda,\beta\delta}(q+p)V^{\nu\delta\gamma}(-p,q+p,-q)D_{\Lambda,\gamma\alpha}(q)]_{reg}. \tag{44}$$

A completely analogous term comes from the coupling with the matter fields (figure 3b) where the gluon propagators are replaced by the quark propagators and a multiplicative factor of -2 is added. Finally there is a contribution from the tadpole graph (figure 3c).

A comment about the ultraviolet regularization is in order here. In principle, due to the renormalizability property, all ultraviolet renormalization are equivalent for the low energy predictions, however in practice some choice is simpler than others. In the following two sections we first discuss the case of a gauge-inconsistent regularization both with elementary diagrammatic methods and with general theoretical methods. The advantages of a gauge-consistent ultraviolet regularization, as for instance dimensional regularization or an higher derivative regularization, are explained in section 7.[10]

# 5 One-loop check of Ward identities

As starting point, let us consider the transversality property of the gluon self-energy

$$p^\mu \Pi^{(1)}_{\mu\nu}(p,n;\Lambda) = 0. \tag{45}$$

If we fix the boundary conditions at the ultraviolet scale $\Lambda_0$ the self-energy is given by the evolution equation (43) as

$$\Pi^{(1)}_{\mu\nu}(p,n;\Lambda) = \Pi^{(1)}_{\mu\nu}(p,n;\Lambda_0) + F^{reg}_{\mu\nu}(p,n;\Lambda) - F^{reg}_{\mu\nu}(p,n;\Lambda_0). \tag{46}$$

[10]Notice that with generic cutoff functions, there is no advantage in using a gauge-consistent ultraviolet regularization since the cutoff $\Lambda$ breaks the symmetry in the *infrared*: it is only with our specific choice that the analysis in section 7 is meaningful.

Now we have to specify the intermediate regularization. For definiteness we regularize the ultraviolet behavior of the (free) propagator by replacing

$$D_{\Lambda,\mu\nu}(q) \to D^{\Lambda_0}_{\Lambda,\mu\nu}(q) = \bar{K}_{0\Lambda_0}(q) D_{\Lambda,\mu\nu}(q), \tag{47}$$

where $\bar{K}_{0\Lambda_0}(q)$ is a suitable ultraviolet cutoff function, for instance

$$\bar{K}_{0\Lambda_0}(q) = e^{-q^2/\Lambda_0^2}. \tag{48}$$

Consider first the contribution from graph 3a. The proof of the transversality property (45) is based on the tree level vertex identity

$$p^\mu V_{\mu\nu\rho}(p, q, -q-p) = D^{-1}_{\Lambda,\nu\rho}(q) - D^{-1}_{\Lambda,\nu\rho}(q+p) \tag{49}$$

which has the same form of the classical identity (10), thanks to our good choice of the cutoff function $Q_{\Lambda,\mu\nu}$. By using (49) we obtain for the transverse part of the self-energy a contribution of kind

$$p^\mu \Pi^{(1)(a)}_{\mu\nu}(p, n; \Lambda) = p^\mu F^{(a)reg}_{\mu\nu}(p, n; \Lambda) - p^\mu F^{(a)reg}_{\mu\nu}(p, n; \Lambda_0) + b.c. \tag{50}$$

where

$$\begin{aligned} p^\mu F^{(a)reg}_{\mu\nu}(p, n; \Lambda) = \frac{ig^2}{2} \int_q & -D^{\gamma\delta}_\Lambda(q) V_{\gamma\nu\delta}(-q, -p, q+p) \\ & \cdot \bar{K}_{0\Lambda_0}(q) \left[\bar{K}_{0\Lambda_0}(q+p) - \bar{K}_{0\Lambda_0}(q-p)\right] \end{aligned} \tag{51}$$

and $b.c.$ denotes the boundary conditions on the transverse part of the self energy at the scale $\Lambda_0$. For dimensional and covariance reasons we can write

$$\begin{aligned} & p^\mu F^{(a)reg}_{\mu\nu}(p, n; \Lambda) - p^\mu F^{(a)reg}_{\mu\nu}(p, n; \Lambda_0) = \\ & c^{(a)}_1 p^2 \; pn \; n_\nu + c^{(a)}_2 p^2 \; p_\nu + c^{(a)}_3 \Lambda^2 pn \; n_\nu + c^{(a)}_4 \Lambda^2 p_\nu + O(\Lambda^2/\Lambda_0^2, p^2/\Lambda_0^2) \end{aligned} \tag{52}$$

where the precise value of the numerical coefficients $c^{(a)}_i$ depends on the choice of the ultraviolet cutoff function $\bar{K}_{0\Lambda_0}(q)$ but it is not important here. The same argument holds when the coupling with the matter fields is considered (figure 3b), since the analogous identity[11]

$$p^\mu \gamma_{\mu\alpha\beta} = S^{-1}_{\Lambda,\alpha\beta}(q) - S^{-1}_{\Lambda,\alpha\beta}(q+p). \tag{53}$$

[11] $S_{\alpha\beta}(p; \Lambda)$ denotes the tree-level quark propagator.

for the gluon-matter-matter vertex $\gamma_{\mu\alpha\beta}$ holds. Therefore this graph gives a contribution $c_i^{(b)}$ to the relevant coefficients plus a contribution of order $O(\Lambda^2/\Lambda_0^2, p^2/\Lambda_0^2)$ to the irrelevant coefficients. The tadpole graph (figure 3c) is momentum independent and gives solely a contribution to the coefficients $c_3^{(c)}$ and $c_4^{(c)}$. The important point is that it is possible to recover the transversality of the self-energy at the scale $\Lambda$ provided that we choose the ultraviolet boundary conditions on the relevant couplings associated to non-transverse terms as

$$p^\mu \Pi^{(1)}_{\mu\nu,rel}(p,n;\Lambda_0) \equiv p^\mu F^{reg}_{\mu\nu,rel}(p,n;\Lambda_0) - p^\mu F^{reg}_{\mu\nu,rel}(p,n;\Lambda). \tag{54}$$

This is the fine-tuning procedure. In this way the non-transverse terms are vanishing when the ultraviolet cutoff $\Lambda_0$ is removed and therefore gauge-invariance is recovered at the end.

Using the classical Ward identities (9)-(11) one could apply the same arguments to all the remaining one-loop Ward-identities. However we prefer to give a general proof based on the Quantum Action Principle.

# 6 General proof of gauge-invariance

The effects of a non-gauge-invariant regularization can be studied in the formalism of the modified Ward identities. In appendix B we prove that the variation of the effective action under linear transformation when a generic cutoff function $R_\Lambda(q)$ is employed has the form

$$\delta\Pi(\Phi;\Lambda) = i\hbar \mathrm{STr} R_\Lambda L (R_\Lambda + \Pi_{\tilde\Phi\Phi})^{-1} - \Delta_\Gamma \tag{55}$$

where $\Delta_\Gamma$ is local (this is the content of the Quantum Action Principle). Therefore the condition $\Delta_\Gamma = 0$ is equivalent to a system of equations for the spurious (i.e. associated to non-invariant operators) relevant couplings which can be solved in perturbation theory. These are the general fine-tuning conditions. In particular we can introduce a class of regular cutoff terms which limit is our specific cutoff,

$$R^{\Lambda_0}_{\Lambda,\mu\nu}(q^2) = \bar K_{0\Lambda_0}(q^2)\left(\Lambda^2 g_{\mu\nu} + \frac{1}{\xi_2}\frac{n_\mu n_\nu}{n^2}\right). \tag{56}$$

A class as (56) has been implicitly used in the one-loop analysis of the gluon self-energy transversality given in the section 5 since it corresponds to a propagator exponentially dumped in the ultraviolet such as (47).

If for notational simplicity we work in the pure Yang-Mills case (the extension to a vector-like coupled matter is trivial) we can rewrite (55) in a more explicit notation as

$$\mathcal{W}_f \Pi = \int_{x,y} i R^{\Lambda_0}_{\Lambda,\mu\nu}(-\partial_x^2)\delta(x-y) f(x) \cdot <0|T A^\mu(x) \times A^\nu(y)|0>_{c,J}, \tag{57}$$

where $<\cdots>_{c,J}$ denotes the connected correlation function in presence of the non-zero source $J_\mu = -\frac{\delta\Gamma}{\delta A^\mu}$. It is clear from this formula that when the ultraviolet cutoff is removed (*after* the correct subtractions in the proper vertices has been done) the right hand side is identically zero simply due to the antisymmetry of the structure constants. Therefore even in this case we can prove

$$\delta\Pi(\Phi;\Lambda) = 0. \tag{58}$$

As a matter of fact, all the results reported in this paper can be generalized, at least at the perturbative level, in a way independent of the ultraviolet regularization, provided that the fine tuning problem is solved. The new point of this paper is the fact that actually it is possible to avoid the fine-tuning procedure by adopting a gauge-consistent regularization, the simplest being dimensional regularization.

# 7 The simple proof of gauge-invariance

If we adopt a gauge-consistent intermediate regularization then gauge-invariance can be directly proved from the functional form of the ERGE [12] [1, 14]

$$\dot{\Pi}(\Phi;\Lambda) = I(\Phi;\Lambda) = \left[-\frac{i}{2}\mathrm{STr}\dot{Q}_\Lambda \left(Q_\Lambda + \Pi_{\tilde{\Phi}\Phi}\right)^{-1}\right]_{reg-inv}, \tag{59}$$

where we have defined

$$\Pi_{\tilde{\Phi}\Phi} = \frac{\delta}{\delta\Phi}\frac{\overleftarrow{\delta}}{\delta\Phi}\Pi(\Phi;\Lambda). \tag{60}$$

The logic of the proof is the following [1].

[12] One could give a third proof of gauge-invariance by deducing (58) by standard manipulations in the functional integral. Actually, this is the less rigorous but simpler approach.

1. We suppose that the functional $\Pi(\Phi;\Lambda)$ is gauge-invariant at some initial scale $\bar{\Lambda}$,

$$\mathcal{W}_f \Pi(\Phi;\bar{\Lambda}) = 0. \tag{61}$$

2. We observe that in this hypothesis even the functional $I(\Phi;\bar{\Lambda})$ is gauge-invariant,

$$\mathcal{W}_f I(\Phi;\bar{\Lambda}) = 0. \tag{62}$$

3. Therefore the evolution equation is gauge-invariant and, as a consequence, the Ward identities are satisfied to any $\Lambda$.

Here we give a very elegant proof, valid for generic linear symmetries[13], i.e. for infinitesimal $\Lambda-$independent field transformations of the kind

$$\delta\Phi^A = L^A_B \Phi^B + \ell^A \tag{63}$$

such as the classical action is invariant

$$\delta S_{CL}(\Phi;\Lambda) = (L\Phi + \ell)^A \frac{\delta S_{CL}}{\delta \Phi^A} = 0. \tag{64}$$

For instance in a pure Yang-Mills theory (but the same property holds even when the coupling with the matter is considered) the gauge symmetry $\delta A_\mu = \mathcal{W}_f A_\mu = -f \times A_\mu + \partial_\mu f$ is linear with

$$L^A_B = -f^a_b(x)\delta(x-y)\delta^\mu_\nu, \quad f^a_b(x) \equiv f^c(x)(\tau_c)^a_b, \quad \ell^A = \partial^\mu f^a(x). \tag{65}$$

In general the quantization procedure can break the symmetry. However, if the cutoff function $Q_\Lambda$ has the exceptional property (which holds in our case)

$$Q_\Lambda \tilde{L} + L Q_\Lambda = 0, \tag{66}$$

then the breaking term is linear in quantum fields

$$\delta\Gamma^{(0)}(\Phi;\Lambda) = \tilde{\ell} Q_\Lambda \Phi \tag{67}$$

[13]Obviously this proof also applies to the Abelian case in covariant gauges and may be seen as an alternative analysis with respect to the diagrammatic proof of [1].

and the evolution equation is gauge invariant. To prove this statement we use the transformation law of $\Pi_{\tilde{\Phi}\Phi}(\Phi;\bar{\Lambda})$ under finite transformations

$$\Phi' = e^L \Phi + \ell \tag{68}$$

(in geometric language, these transformations are affine isometries with respect to the metric in the fields space defined by $Q_{\Lambda,AB}$) which is simply

$$\Pi_{\tilde{\Phi}\Phi}(\Phi';\bar{\Lambda}) = e^L \Pi_{\tilde{\Phi}\Phi}(\Phi;\bar{\Lambda}) e^{\tilde{L}}. \tag{69}$$

By using the explicit form of $I(\Phi;\bar{\Lambda})$ and the invariance property of $Q_\Lambda$, which follows from (66)

$$e^L Q_\Lambda e^{\tilde{L}} = Q_\Lambda, \tag{70}$$

one immediately prove the gauge invariance of the evolution equation at the scale $\bar{\Lambda}$

$$I(\Phi';\bar{\Lambda}) = I(\Phi;\bar{\Lambda}), \tag{71}$$

therefore at all scales.

# 8 Gauge-dependence and unitarity

In order to give a rigorous status to the axial gauge formulation, one should prove the following statements:

1. Physical amplitudes are consistent with unitarity.

2. Physical quantities are independent of the gauge vector $n_\mu$.

The fulfillment of these properties is non-trivial in the usual axial gauge formulation, due to the difficult problem of spurious divergences. In particular a canonical analysis shows that the unitarity issue is very subtle [21]. In a generic Wilsonian framework [6] the spurious divergences are avoided, but there is an explicit unitarity breaking at $\Lambda \neq 0$ due to the breaking of Ward identities. Fortunately, the situation is better in our formalism and we can easily prove the consistency with unitarity for any finite $\Lambda \neq 0$ [33]. However, we will lose covariance for any $\Lambda \neq 0$.

To show unitarity, we have to prove that our propagator (28) is consistent with the Landau-Cutkosky rules. Let us consider an orthonormal basis $e^{(\mu)}_\lambda(p,n)$ satisfying

$$e^{(\mu)}_\lambda e^{(\nu)*}_\rho g^{\lambda\rho} = g^{\mu\nu}, \quad e^{(\mu)}_\lambda e^{(\nu)*}_\rho g_{\mu\nu} = g_{\lambda\rho} \tag{72}$$

and let us take the vector $P_\mu = p_\mu - n_\mu pn/n^2$ which is on-shell time-like, i.e. $P^2 > 0$ for $p^2 = \Lambda^2$. If we define

$$e^{(0)}_\mu = P_\mu/\sqrt{P^2}, \quad e^{(3)}_\mu = n_\mu/\sqrt{-n^2}, \tag{73}$$

then the property

$$\sum_{i=1}^{2} e^{(i)}_\mu e^{(i)*}_\nu = -g_{\mu\nu} + \frac{-pn(n_\mu p_\nu + p_\mu n_\nu) + n^2 p_\mu p_\nu + p^2 n_\nu n_\nu}{p^2 n^2 - (pn)^2} \tag{74}$$

holds. As a consequence we can write

$$\mathrm{Im} D_{\Lambda,\mu\nu}(p) = \pi\delta(p^2 - \Lambda^2)\theta(p_0)\sum_{i=1}^{2} e^{(i)}_\mu e^{(i)*}_\nu \tag{75}$$

and this assures the consistency with the optical theorem and unitarity. Therefore apparently we have constructed an unitary massive renormalizable non-Abelian gauge theory. Nevertheless, this theory is unphysical at $\Lambda \neq 0$ due to an incurable breaking of Lorentz covariance.

To understand this point we consider, as a specific example, the gauge-invariant quantity

$$G(p,n) = \int_x e^{ipx} < 0|T[F_{\mu\nu}F^{\mu\nu}](x)[F_{\mu\nu}F^{\mu\nu}](0)|0 > . \tag{76}$$

At lowest order in perturbation theory this quantity is given by a Feynman diagram containing the transverse part of the axial gauge propagator,

$$\begin{aligned} -D^T_{\mu\nu}(p;\Lambda) &= -t_{\mu\rho}(p)D^{\rho\sigma}(p;\Lambda)t_{\sigma\nu}(p) \\ &= \frac{t_{\mu\nu}(p)}{p^2 - \Lambda^2 + i\varepsilon} + \frac{\Lambda^2 N_\mu N_\nu}{((p\cdot n)^2 - n^2\Lambda^2)(p^2 - \Lambda^2 + i\varepsilon)}, \end{aligned} \tag{77}$$

with $N_\mu = n_\mu - p_\mu pn/p^2$. From the explicit formula one sees that at $\Lambda = 0$ the $n_\mu-$dependence cancels: however for $\Lambda \neq 0$ even gauge-invariant observables *does* depend on the quantization direction $n_\mu$. Nevertheless, in the ultraviolet region the $n_\mu$-dependence, i.e. the explicit Lorentz-covariance breaking term, is suppressed. Therefore the Wilsonian approach can give reasonable results even when $\Lambda$ is not exactly zero. In other words we expect the $\Lambda-$dependence be analytical for a certain physical observables in some momentum range and for reasonable approximation schemes. This point is of crucial importance because in numerical simulations $\Lambda$ is never completely removed. However we notice that there are situations where the naïve $\Lambda \to 0$ limit does not exists; and the subtle problems connected with this limit will be discussed in a forthcoming paper [5].

Here we are more interested in the Lorentz-covariance issue. We want to provide a machinery to control the Lorentz-covariance breaking terms. The method we use is based on BRST techniques and directly inspired to analysis of [22].

Consider first the pure Yang-Mills case (the analysis trivially generalize to a vector-like matter coupling) at $\Lambda = 0$. As it is well known [2], even in the axial gauge framework one can introduce the ghosts $C^a$ and $\bar{C}^a$ and the auxiliary field $\lambda^a$ with BRST transformations

$$sA = DC, \quad sC = -\frac{1}{2} g C \times C, \quad s\bar{C} = \lambda, \quad s\lambda = 0 \tag{78}$$

and one can take as tree level action[14]

$$S_{BRST} = S_{CL}(A) + \int_x s\left(\bar{C} n_\mu A^\mu\right) = S_{CL}(A) + \int_x n^\mu \left(A_\mu \lambda - \bar{C} D_\mu C\right). \tag{79}$$

Now it is clear from (79) that the quantization direction $n_\mu$ cannot influence physical quantities, since it couples with a BRST-trivial operator. However, if the infrared cutoff term $\frac{1}{2}\Lambda^2 A_\mu \cdot A^\mu$ is considered, the BRST symmetry is explicitly broken,

$$s\left(S_{BRST} + \int_x \frac{1}{2}\Lambda^2 A_\mu \cdot A^\mu\right) = \int_x \Lambda^2 A_\mu \partial^\mu C \neq 0.$$

Nevertheless, there is a way to control the BRST breaking term and the $n_\mu$-dependence by extending the BRST-symmetry in a suitable way. To this

[14] In general axial gauges one adds the BRST-trivial operator $s\left(\bar{C} \cdot nA - \frac{\xi_2}{2}\bar{C} \cdot \lambda\right) = n_\mu A^\mu \cdot \lambda - \bar{C} n^\mu D_\mu C - \frac{\xi_2}{2}\lambda^2$.

Table 1: Pure Yang-Mills theory.

| Field or parameter | Mass dimension | Ghost number |
|---|---|---|
| $A$ | 1 | 0 |
| $C$ | 0 | 1 |
| $\bar{C}$ | 3 | -1 |
| $\lambda$ | 3 | 0 |
| $n_\mu$ | 0 | 0 |
| $\varphi_\mu$ | 0 | 1 |
| $\bar{\rho}$ | 2 | -1 |
| $A^*$ | 3 | -1 |
| $C^*$ | 4 | -2 |

aim we introduce two Grassmann constants $\bar{\rho}$ and $\varphi_\mu$ with mass dimension and Grassmann number respectively 2,-1 and 0,1 (see table 1) and extended BRST-transformation

$$s_E\ \bar{\rho} = \Lambda^2, \quad s_E\ \varphi_\mu = 0; \tag{80}$$

moreover we give to the gauge vector the transformation

$$s_E\ n_\mu = \varphi_\mu; \tag{81}$$

on the quantum fields $s_E$ exactly coincides with $s$. Now the extended action

$$\begin{aligned} S_{EXT}(\Phi; \Lambda^2, \varphi_\mu, \bar{\rho}) &= S_{CL}(A) + s_E \left( \bar{C} \cdot n_\mu A^\mu + \frac{1}{2} \bar{\rho} A_\mu \cdot A^\mu \right) \\ &= S_{CL}(A) + \int_x nA \cdot \lambda - \bar{C} \cdot n^\mu D_\mu C - \bar{C} \cdot \varphi_\mu A^\mu \\ &\quad + \frac{1}{2} \Lambda^2 A \cdot A - \bar{\rho} A^\mu \cdot \partial_\mu C \end{aligned} \tag{82}$$

is invariant under the extended BRST-symmetry and all the usual techniques can be used to lift this symmetry to the quantum level in the form of a generalized Slavnov-Taylor equation which reads

$$\int_x \left( \frac{\delta \Gamma^{(0)}}{\delta A^*} \cdot \frac{\delta \Gamma^{(0)}}{\delta A} + \frac{\delta \Gamma^{(0)}}{\delta C^*} \cdot \frac{\delta \Gamma^{(0)}}{\delta C} + \lambda \cdot \frac{\delta \Gamma^{(0)}}{\delta \bar{C}} \right) + \varphi_\mu \frac{\partial \Gamma^{(0)}}{\partial n_\mu} + \Lambda^2 \frac{\partial \Gamma^{(0)}}{\partial \bar{\rho}} = 0. \tag{83}$$

In (83), as usual, we have introduced two external sources $A^*_\mu$ and $C^*$ coupled to the non-linear variation $sA_\mu$ and $sC$, respectively. The perturbative machinery allows to extend this identity to all orders in the loop-wise expansion[15] since no anomalies are generated. Now by deriving with respect to $\varphi_\mu$ and by putting $\varphi_\mu = 0$ one obtains the functional dependence on the gauge vector $n_\mu$. In particular in the $\Lambda \to 0$ limit one proves the gauge-independence of physical quantities [22]. Moreover in $\Lambda \to \infty$ limit, when the effective action becomes local and has the *same* form of the (extended) BRST-action, the symmetry forbids the presence of the $h(\Lambda)$ term introduced in (38). In other words, $h^{(\ell)}(\Lambda)$ must be subleading with respect to the other relevant couplings and in particular at one-loop is finite (actually $h^{(1)}(\Lambda) \equiv 0$ since the boundary condition $h(\Lambda_R) = 0$ is imposed).

The important point we wish to stress here, is that the axial BRST symmetry is much simpler than the covariant usual one: this is due to the fact that the additional fields $(\lambda, \bar{C}, C)$ we have introduced *decouple* completely from the theory. This is due respectively to the constraint equation

$$\frac{\delta \Gamma^{(0)}}{\delta \lambda} = \Delta^{(0)}_\lambda = n^\mu A_\mu, \tag{84}$$

the ghost equation

$$\frac{\delta \Gamma^{(0)}}{\delta \bar{C}} - gC \times \frac{\delta \Gamma^{(0)}}{\delta \lambda} = \Delta^{(0)}_{\bar{C}} = -n^\mu \partial_\mu C - \varphi^\mu A_\mu \tag{85}$$

and finally the anti-ghost equation

$$\frac{\delta \Gamma^{(0)}}{\delta C} - g\bar{C} \times \frac{\delta \Gamma^{(0)}}{\delta \lambda} = \Delta^{(0)}_C \tag{86}$$

with

$$\Delta^{(0)}_C = -n^\mu \partial_\mu \bar{C} - \bar{\rho} \partial_\mu A^\mu + \partial^\mu A^*_\mu - gA^* \times A - gC^* \times C. \tag{87}$$

[15] In the Wilsonian point of view, one defines

$$\int_x \left( \frac{\delta \Gamma}{\delta A^*} \cdot \frac{\delta \Gamma}{\delta A} + \frac{\delta \Gamma}{\delta C^*} \cdot \frac{\delta \Gamma}{\delta C} + \lambda \cdot \frac{\delta \Gamma}{\delta \bar{C}} \right) + \varphi_\mu \frac{\partial \Gamma}{\partial n_\mu} + \Lambda^2 \frac{\partial \Gamma}{\partial \bar{\rho}} = \Delta_\Gamma$$

and checks the consistency with the evolution equation, by showing that $\Delta_\Gamma = 0 \Longrightarrow \dot{\Delta}_\Gamma = 0$. Technically is simpler to study the functional $\Delta_W(J) = \Delta_\Gamma(\Phi)$, as done in appendix B. In the proof the identity $-\frac{\partial W}{\partial \bar{\rho}} = \int_x \frac{\delta W}{\delta A^*_\mu \delta J^\mu} + i \frac{\delta W}{\delta A^*_\mu} \frac{\delta W}{\delta J^\mu}$ is helpful.

These functional identities correspond to (non-anomalous) linearly broken linear symmetries and can be extended to all orders in perturbation theory. In particular the right hand side does not renormalize. Notice that this fact is independent of the specific adopted renormalization procedure, since the general analysis described in appendix B and based on the QAP applies. Therefore the crucial features of the axial gauge formulation are maintained even at $\Lambda \neq 0$.

# 9 Beta function computation

As we explained in full detail in [1] one can extract from the relevant part of the Wilson Renormalization Group Equation the Callan-Symanzik renormalization group functions i.e. the beta function and the anomalous dimension. In this section we present, as a non-trivial consistency check of the formalism, a one-loop computation of the QCD beta function. Here we will restrict to the pure gauge part, since the contribution from the matter fields is exactly the same as in covariant gauge. This example will be worked out in detail, since the computation is quite different with respect to more standard analysis.

As it is well known, one of the technical advantages of the axial gauge formulation is the fact that the beta function can be directly computed from the gluon self-energy, thanks to the QED-like identity (42) which relates the beta function to the anomalous dimension $\gamma_A(g)$,

$$\beta(g) = g\gamma_A(g), \quad \gamma_A(g) = -\frac{1}{2}\frac{\dot{Z}_A}{Z_A}. \tag{88}$$

Taking in account the fact that, as a consequence of the generalizated BRST symmetry, the relation $\dot{h}^{(1)}(\Lambda) = 0$ holds (this can also be checked with an explicit computation), we can directly extract the wave function $\dot{Z}_A^{(1)}$ from the gluon self-energy. By using the choice $n^\mu = (0,0,0,1)$ the relevant integral comes from the 33-component of the self-energy,

$$\dot{Z}_A^{(1)} = \partial_{\bar{p}^2}\dot{\Pi}_{33}^{(1)}|_{p=0}, \quad \bar{p} = (p^0, p^1, p^2). \tag{89}$$

Due to the simple form of the (Euclidean) gluon propagator

$$\begin{aligned} D_{\Lambda,33} &= D_{\Lambda,3i} = D_{\Lambda,i3} = 0, \quad i = 0,1,2 \\ D_{\Lambda,ij} &= \frac{\delta_{ij}}{p^2+\Lambda^2} + \frac{p_i p_j}{(p^2+\Lambda^2)(p_3^2+\Lambda^2)} \end{aligned} \tag{90}$$

and the explicit form of the trilinear vertex, there is a big simplification in the integral defining $\dot{Z}_A^{(1)}$ with respect to the general integral (43). Moreover, by using the trick reported in appendix C equation (102) one can reduce this integral to an even simpler integral in $\bar{q}$ evaluated at $\Lambda = 0$ which explicit form is

$$\dot{Z}_A^{(1)} = -g^2 N_c \int_{\bar{q}} \frac{4}{3} \, \frac{3\ q_3^6 + 2\ q_3^2\ \bar{q}^4 + \bar{q}^6}{(\bar{q}^2 + q_3^2)^4\ q_3\ \pi}. \tag{91}$$

This integral can be computed by using equation (105) and finally the expected result

$$\beta^{(1)}(g) = -g\dot{Z}_A^{(1)}/2 = -\frac{1}{16\pi^2}\frac{11}{3}N_C\ g^3 \tag{92}$$

is obtained.

We remark that the computation is actually *independent* of the cutoff function choice. Therefore the same result can be obtained even in other approaches, when generic cutoffs are employed. This feature is due to two reasons:

1. The identity $\beta^{(1)} = g\gamma_A^{(1)}$ holds even if the gauge symmetry is broken (but *only* at one-loop). This a consequence of the Quantum Action Principle.

2. The relation (102) in appendix C allows to relate an integral apparently dependent on the infrared cutoff $\Lambda$ to an universal integral independent of the cutoff function choice.

# 10 Conclusions

In this paper we have solved the problem of giving an explicitly gauge and unitarity consistent Wilsonian formulation of non-Abelian gauge theories. The price to pay is the lost of covariance, i.e. there is an unphysical dependence on the gauge vector $n_\mu$ for $\Lambda \neq 0$. However this dependence vanishes in the $\Lambda \to 0$ limit for physical quantities (assuming we consider a quantity such as the $\Lambda \to 0$ limit exists) whereas for finite $\Lambda$ is controlled by a simple generalized Slavnov-Taylor identity. Therefore there is a strong progress with respect to generic Wilsonian procedures, where gauge-invariance and unitarity are lost and the gauge-dependence is much more difficult to study.

In the approach we presented, one not only maintains all the expected advantages both of the non-covariant gauge formulation (simple Ward identities and decoupling of the ghosts) and of the Wilsonian viewpoint (simple understanding of the renormalizability issue and of the relation with the field theory renormalization group), but there are also additional bonus: in particular spurious divergences are automatically avoided and we have efficient methods to perturbatively compute beta functions and anomalous dimensions. However, the greatest potentiality of this approach relies on the possibily of starting a non-perturbative analysis via suitable *gauge-invariant* truncations, thus solving a major problem of the usual ERGE approach to gauge theories [27, 28].

An important point which has not been addressed in this paper, is the question of the infrared limit $\Lambda \to 0$. This point is very delicate. As a matter of fact, in the usual formulation with the CPV prescription, various inconsistencies have been found [4]. These problems of principle of the axial gauge choice will stay also in the Wilsonian approach in a disguised form, as we will discuss in [5]. Probably the simpler way to avoid these problems is to switch to planar or light-cone gauges, which are safe in the usual formalism. This will be the subject of [5]. However, the main results of this paper i.e. the consistency with Ward identities and the control of the Lorentz-covariance breaking (i.e. the $n_\mu-$dependence) straightforwardly generalize to all non-covariant gauges.

Moreover, it is straightforward to extend the part of this work concerning the (extended) BRST symmetry to the covariant gauge case. Actually it is possible to control the BRST breaking mass term and the gauge-dependence issue by using the same tricks we have used in this paper. However in this case unitarity is lost at $\Lambda \neq 0$, ghosts do not decouple, and the analysis of Slavnov-Taylor identities is less simple. The obvious advantage is that the manifest covariance is maintained. In general the problem of how non-linear symmetries are treated in our version of the Wilson Renormalization Group is an interesting issue which should be investigated in the future. The only non-trivial point is that the definition of relevant couplings in terms of zero-momentum subtractions is inconsistent with non-linear symmetries and cannot be maintained. Instead some minimal subtraction similar to the dimensional regularization procedure must be invoked.

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# A Sketch of the renormalizability proof

For sake of completeness, in this appendix we briefly comment about the renormalizability property in the axial gauge framework. The point is that the Wilsonian renormalizability proof stated in [1] for a large class of cutoff functions can be extended to the present case. Here we simply sketch the argument, a rigorous proof can be easily given by following the lines described here and the work done in [1].

Technically speaking, the great advantage of the Wilsonian formulation is the fact that, once one has proved that the integrated evolution equation is well defined when the ultraviolet cutoff is removed at the first order of iteration, then this property is immediately transported to all orders in perturbation theory. In other words, no analysis of overlapping divergences is required [13] to prove that the proper vertices are well defined in the $\Lambda_0 \to \infty$ limit.

In the case at hand, one sees that all the one-loop Feynman diagrams in the integrated evolution equation are convergent by direct inspection. This is straightforward since in the renormalizability proof the infrared cutoff $\Lambda$ is different from zero and therefore the spurious divergences are automatically avoided. In particular for $n^\mu = (0, 0, 0, 1)$, one can split the four dimensional integration $\int_q$ as $\int_{q^3} \int_{\bar{q}}$. All the three-dimensional integrals are made finite by using the natural prescription reported in appendix C; therefore one must only control the $q_3$ integrals which are convergent for dimensional analysis, due to the momentum subtractions.

In a more conventional language the renormalizability is expected since in the axial gauge formalism there is a power counting criterium and the ultraviolet divergences are local polynomials in the momenta [2].

## B Formal proof of the mWI and relation with the QAP

In various part of the text we have used some results from the Quantum Action Principle (QAP). For completeness, here we review its proof in the Wilson Renormalization Group formulation, by focusing in particular on the analysis of (possibly linearly broken) linear symmetries. The source of this kind of rigorous proofs can be found in the work of Becchi [10]. A translation in the cutoff action formalism is given in [9] (see also [8]). Both references consider the non-linear symmetries, but the analysis is essentially the same also for linear symmetries.
Let us consider the general modified Ward identity (mWI)

$$-\delta\Pi + i\hbar STrR_\Lambda L(R_\Lambda + \Pi_{\tilde{\Phi}\Phi})^{-1} = \Delta_\Gamma , \tag{93}$$

where $\hbar$ has been explicited since our analysis will be perturbative in the loop-wise expansion. The QAP [11] says that if $\Delta_\Gamma^{(\ell')} = 0$ for $\ell' = 0, \dots, \ell$, then $\Delta_\Gamma^{(\ell+1)}$ is local, i.e. contains a finite number of relevant terms. Under certain conditions, i.e. in absence of anomalies, this fact assures that it is possible to impose

$$\Delta_\Gamma^{(\ell)}(\Phi; \Lambda) = 0 \tag{94}$$

to all orders. Technically, the proof proceeds by showing that the functional identity (94) is consistent with the evolution equation. Therefore if it holds at the initial scale $\bar{\Lambda}$ it holds at all scales. The proof is independent of the used cutoff function $R_\Lambda(q)$, which can be at large extent generic.

To prove the statement it is convenient to study the functional

$$\Delta_W(J; \Lambda) = \left(\tilde{J} + \frac{\overleftarrow{\delta}\, W}{\delta J} R_\Lambda\right)\left(L\frac{\delta W}{\delta J} + \ell\right) - i\hbar \mathrm{STr} R_\Lambda L \frac{\delta}{\delta J}\frac{\overleftarrow{\delta}\, W}{\delta J} . \tag{95}$$

Notice that $\Delta_\Gamma(\Phi; \Lambda) = \Delta_W(J; \Lambda)$ is zero *iff* $\Delta_W(J; \Lambda)$ is zero. To prove this latter fact we use the evolution equation of the $W(J; \Lambda)$ functional [1]

$$\dot{W} = -\frac{i}{2}\mathrm{STr}\dot{R}_\Lambda \left(\hbar \frac{\delta}{\delta J}\frac{\overleftarrow{\delta}\, W}{\delta J} + i\frac{\delta W}{\delta J}\frac{\overleftarrow{\delta}\, W}{\delta J}\right) . \tag{96}$$

By a lengthy but straightforward computation one obtains the evolution equations of $\Delta_W$ which is *linear* of the kind

$$\dot{\Delta}_W = M_W \Delta_W \tag{97}$$

where $M_W$ is the functional operator

$$M_W = -\frac{i}{2}\mathrm{STr}\dot{R}_\Lambda\left(\hbar\frac{\delta}{\delta J}\frac{\overleftarrow{\delta}}{\delta J} + i\frac{\delta W}{\delta J}\frac{\overleftarrow{\delta}}{\delta J} + i\frac{\delta}{\delta J}\frac{\overleftarrow{\delta}\, W}{\delta J}\right). \tag{98}$$

Therefore if $\Delta_W$ is zero at some scale $\bar{\Lambda}$, it is identically zero and the functional identity (94) holds. Equation (97) has been introduced and solved in [10]; however it is more convenient to consider the equivalent identity [9]

$$\Lambda\partial_\Lambda \Delta_\Gamma = \hbar M_\Gamma \Delta_\Gamma \tag{99}$$

where

$$M_\Gamma = \frac{i}{2}\mathrm{STr}(R_\Lambda + \Pi_{\tilde{\Phi}\Phi})^{-1}\dot{R}_\Lambda(R_\Lambda + \Pi_{\tilde{\Phi}\Phi})^{-1}\frac{\delta}{\delta\Phi}\frac{\overleftarrow{\delta}}{\delta\Phi}. \tag{100}$$

Equation (97) is simpler since can be solved recursively in $\hbar$ and can be used to give an easy proof of the QAP in the Wilsonian formalism. We remind that the QAP is the basic building block of the algebraic renormalization program [31] and is conceptually very important: in particular a renormalization scheme should be considered viable only if it is consistent with this principle[16]. Fortunately, the proof of the QAP is very simple in the Wilson Renormalization Group approach, at least at the perturbative level. In fact from (99), by using the fact that $\Delta_\Gamma^{(0)}$ vanishes (i.e. the symmetry holds at the classical level) one obtains that $\Delta_\Gamma^{(1)}$ is $\Lambda-$independent. Therefore it is equal to its value at the ultraviolet scale $\Lambda_{UV} >> \Lambda_R$. But at this scale $\Delta_\Gamma^{(1)}$ is local for dimensional reasons (the irrelevant parts are suppressed as inverse powers of $\Lambda_{UV}$) therefore the QAP holds at one loop: the breaking term is local. Now, may be possible to impose $\Delta_\Gamma^{(1)} = 0$ at the scale $\Lambda_{UV}$, depending on the algebraic structure of the theory, i.e. on the study of the Wess-Zumino consistency conditions [31]. If the structure is trivial (i.e. there

[16]For instance, we remind that dimensional reduction in supersymmetric theories is a largely used but inconsistent renormalization scheme [32].

are no anomalies) we can impose $\Delta_\Gamma^{(1)} = 0$. This involves a non-trivial fine-tuning of spurious couplings in terms of the physical couplings which explicit form is given by the one-loop relevant part of equation (93). At this point one can iterate the argument and use $\Delta_\Gamma^{(1)} = 0$ and (99) to prove that $\Delta_\Gamma^{(2)}$ is $\Lambda-$independent and therefore local: thus by a new fine-tuning one can impose $\Delta_\Gamma^{(2)} = 0$. The argument obviously generalize to all orders in the loop-wise expansion. As a consequence the mWI (93) holds for any $\Lambda$.

Notice that exactly the *same* analysis holds in the case of theories which are linearly broken at three level. In fact, due to the explicit form of the $M$ operator, which contains second order functional derivatives in the fields, one directly see that all the perturbative corrections $\Delta_\Gamma^{(\ell)}$, $\ell \geq 1$, identically vanishes. Therefore we recover in this context the well known results that linear breaking terms does not renormalize. In particular this remark applies to the functional identities (84), (85) and (86) in the text, as well as to the Ward identities (55), and is completely general, i.e. independent on the used renormalization procedure, provided that the QAP holds and the theory is anomaly free.

In general the previous analysis is required whenever one is interested in theories where the ERGE is inconsistent with the classical symmetries. However in various cases it is impossible to impose $\Delta_\Gamma = 0$. For instance in chiral gauge theories[17] the right hand side of the mWI (93) is non-zero and one could compute the chiral anomaly by following the lines of [29]. Obviously one obtains the same result since the coefficient of the anomaly is independent of the cutoff function choice.

The important point of this paper is the fact that we were able to avoid such a non-trivial analysis in non-anomalous theories like QCD. In fact, by using the Wilsonian renormalization procedure we have presented, supplemented with a gauge-consistent intermediate regularization, the tree-level Ward identities are never broken and therefore $\mathcal{W}_f \Pi = 0$ identically holds, *without fine-tuning.*

# C Technical remarks

In this appendix we collect some formulae helpful in perturbative computations.

[17] In our formalism the ERGE is inconsistent with chiral symmetry due to the spinor "mass" term $i\Lambda\bar{\psi}\gamma_5\psi$.

First we present an useful trick allowing to compute with a little effort dimensionless coefficients of the kind

$$c = \int_{q_E} \Lambda \partial_\Lambda F(q_E; \Lambda) \tag{101}$$

where $F(q_E; \Lambda)$ ha mass dimension $-4$ and is obtained from some Feynman diagram at zero momentum, possibly derived with respect to the external momenta. Typically $c$ is a renormalization group coefficient, in particular the one-loop beta function coefficient. The trick we present[18] allows to rewrite equation (101) in the form

$$c = -\int_{\bar{q}} \frac{q_3}{\pi} F(\bar{q}, q_3; 0) \tag{102}$$

which is simpler since the integration in $q_3$ and the $\Lambda-$dependence disappeared. Notice that since $c$ is dimensionless, the $q_3$-dependence is apparent and cancels, i.e. $\int_{\bar{q}} F(\bar{q}, q_3; 0) \sim 1/q_3$. The derivation of equation (102) is based on the relation $\Lambda\partial_\Lambda f(q_3/\Lambda) = -q_3 \partial_{q_3} f(q_3/\Lambda)$ from which one obtains

$$\int_{q_E} \Lambda \partial_\Lambda F = 2 \int_0^\infty \frac{dq_3}{2\pi q_3} \Lambda \partial_\Lambda \int_{\bar{q}} q_3 F(\bar{q}, q_3; \Lambda) = -\int_{\bar{q}} \frac{q_3}{\pi} F(\bar{q}, q_3; 0),$$

where the property $F(\bar{q}, q_3; \Lambda \to \infty) = 0$ has been used.

The tree-dimensional integrals appearing in (102) can be evalued with standard formulae.

- Formulae for three-dimensional angular averages.

  If $m = 2n$ is even, we have

  $$< (\vec{q} \cdot \vec{p})^m >_{\Omega_3} \equiv \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta (qp\cos\theta)^{2n} = \frac{(qp)^{2n}}{2n+1}, \tag{103}$$

  otherwise, if $m = 2n+1$ is odd, we have

  $$< (\vec{q} \cdot \vec{p})^m >_{\Omega_3} \equiv 0. \tag{104}$$

[18] A similar relation holds in covariant computations and has been proved in the appendix of [30].

- Formulae for the momentum integration.

  Let be $N > 0$, $M \geq 0$ integers and $A > 0$ a $k-$independent real number; for $N > M + 3/2$ the relation

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^{2M}}{(k^2 + A)^N} = \frac{1}{4\pi^2} \frac{B(N - M - 3/2, M + 3/2)}{A^{N-M-3/2}}, \qquad (105)$$

  where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the Euler's Beta function, holds. The relation (105) can be continued to $N < M + 3/2$, by implicitly defining a (natural) prescription for three-dimensional divergent integrals[19]. However in this case the integral is positive only if $-2n < N - M - 3/2 < -2n + 1$, with $n = 1, 2, 3, \ldots$. This observation explains the origin of the negative sign for the beta function in our computation.

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