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Beta function and flowing couplings in the exact Wilson renormalization group in Yang–Mills theory ⋆

M. Bonini [a], G. Marchesini [b], M. Simionato [a]

[a] *Dipartimento di Fisica, Università di Parma, and INFN, Gruppo Collegato di Parma, Italy*

[b] *Dipartimento di Fisica, Università di Milano, and INFN, Sezione di Milano, Milan, Italy*

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Abstract

We discuss the relation between the Gell-Mann–Low beta function and the "flowing couplings" of the Wilsonian action $S_\Lambda[\phi]$ of the exact renormalization group (RG) at the scale Λ. This relation involves the ultraviolet region of Λ so that the condition of renormalizability is equivalent to the Callan–Symanzik equation. As an illustration, by using the exact RG formulation, we compute the beta function in Yang–Mills theory to one loop (and to two loops for the scalar case). We show that the flowing couplings tend to the running coupling at high frequency, they differ at low frequency and remain finite all the way down to zero frequency. We show also that, by a systematical resummation of higher order corrections, the flowing couplings enter directly into the Feynman diagrams with a scale given by the internal loop momenta.

1. Introduction

In renormalization group (RG) there are two different ways to introduce a momentum scale. The first leads to the Gell-Mann–Low beta function and Callan–Symanzik equation, the second to the exact Wilson RG flow. In the first formulation one introduces the scale μ at which the coupling g, mass and wave function normalization are measured or fixed. The fact that physical quantities do not depend on the scale μ leads to the Gell-Mann–Low beta function and the Callan–Symanzik equation. In this formulation

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one does not usually attribute any meaning to the bare action. In the Wilson RG formulation [1] instead the starting point is the local bare action $S_{\Lambda_0}[\phi]$ in which the fields ϕ have frequencies smaller than the ultraviolet (UV) scale Λ_0. The UV action can be viewed as the result of taking a more elementary theory and integrating all fields with frequencies larger than Λ_0. This gives the coefficients of $S_{\Lambda_0}[\phi]$ such as the wave function normalizations $Z_i(\Lambda_0)$ and the parameters $\rho_i(\Lambda_0)$. By further integrating the fields with frequence larger than $\Lambda < \Lambda_0$ one obtains a new action $S_\Lambda[\phi]$ which is not any more local. This Wilsonian action, and in particular the coefficients $\rho_i(\Lambda)$ and $Z_i(\Lambda)$ of its local part, the "relevant parameters", depends on Λ according to the exact RG equation [1–4]. The physical observables are obtained by performing the path integration over all fields with momenta in the full range $0 < q^2 < \Lambda_0^2$. Therefore, for $\Lambda = 0$ (and $\Lambda_0 \to \infty$) the Wilsonian action generates the physical Green functions.[1] In particular for the wave function normalization one has $Z_i(0) = 1$ and the parameters $\rho_i(0)$ are given by the masses and physical coupling g (at the scale μ).

In this paper we establish the relation between the two RG formulations. We analyze, as examples, the cases of the massless scalar and the $SU(2)$ Yang–Mills theories. To stress the difference on the way the scale is introduced, hereafter we shall call μ-RG and Λ-RG the first and the second formulation.

First we deduce the Gell–Mann–Low beta function in terms of the derivative of the relevant parameters $\rho_i(\Lambda)$ and $Z_i(\Lambda)$ of the Wilsonian action. Since RG is related to the UV properties, as expected, this relation involves the UV region of Λ. Although all perturbative contributions to the relevant parameters diverge logarithmically for large Λ, in the combination giving the beta function all divergences cancel.[2] It is interesting to observe that in the Yang–Mills case the UV action in general does not satisfy the BRS symmetry and then it could be surprising that the beta function is given in terms of the UV parameters. However, one should observe that these UV parameters depend on g and μ/Λ_0 in such a way that the physical effective action ($\Lambda = 0$ and $\Lambda_0 \to \infty$) satisfies Slavnov–Taylor identities [4,6].

As a second point concerning the relation between μ-RG and Λ-RG formulations we show that the Callan–Symanzik equation is equivalent to the condition that the theory is renormalizable, i.e. physical vertices do not depend on the UV scale as $\Lambda_0 \to \infty$. This is a special case of the Λ-RG equation which is based on the fact that the physical vertices are obtained from the Wilsonian action $S_\Lambda[\phi]$ independently of the value of Λ.

The natural couplings in Λ-RG are the *Wilsonian flowing couplings* $\hat{g}_i(\Lambda)$ given by the relevant parameters $\rho_i(\Lambda)$ rescaled by the appropriate wave functions $Z_j(\Lambda)$. We show that it is possible to set up an iterative procedure to solve the Λ-RG equations in such a way that one generates contributions of Feynman diagrams in which the loop momenta k_i are ordered in virtuality ($k_1^2 < \ldots k_\ell^2$) and in which the integrands

[1] In the limit $\Lambda_0 \to \infty$ all quantities are perturbatively finite for any fixed Λ (renormalizability). For this reason we often omit to write the Λ_0-dependence in $S_\Lambda[\phi]$ and relevant parameters and, unless explicitly stated, we assume $\Lambda_0 \to \infty$.

[2] In Ref. [5] the beta function has been obtained to one-loop order in terms of the relevant parameters of the Λ-RG formulation.

involve the flowing couplings $\hat{g}_i(k_j)$ at these momentum scales. The flowing couplings themselves can be obtained by a generalized beta function which can be computed as an expansion in the Wilsonian coupling itself. We explicitly compare the Wilsonian and the running couplings in the one-loop resummation.

At large scales the running $g(k)$ and the Wilsonian $\hat{g}_i(k)$ couplings differ only by μ/k power corrections. Therefore increasing k all couplings vanish in the Yang–Mills case, while in the scalar case they grow and become singular at the UV Landau pole $k_L \simeq \mu \exp(1/b_0 g)$ with b_0 the (positive) one-loop coefficient of the beta function. Therefore also within this Λ-RG analysis one finds the property of triviality.

For a k comparable to μ the Wilsonian couplings differ from the running coupling. In the non-Abelian case the one-loop running coupling $g(k)$ diverges at the IR Landau pole and the integration over the small k region become meaningless. In the Λ-RG formulation instead the Wilsonian couplings $\hat{g}_i(k)$ remain finite in all the range of k and at $k = 0$ one has $\hat{g}_i(0) = g$. Therefore the integration over k is now possible.

In the Λ-RG formulation Λ plays the role of an IR cutoff in Feynman diagrams. So, as long as Λ is much larger than μ one can use the perturbative expressions for the various flowing couplings $\hat{g}_i(k_i) \simeq g(k_i)$. The Landau singularity is cutoff by the IR scale Λ. For Λ approaching the physical point $\Lambda = 0$ one has that the various flowing couplings $\hat{g}_i(k_i)$ depart from the running coupling and power corrections resum to make the flowing couplings finite.

The paper is organized as follows. The details of our analysis are presented in Section 2 in the case of the massless scalar theory. In Section 3 we generalize this analysis to the Yang–Mills theory. Section 4 contains some conclusions. In Appendix A we recall the form of the Λ-RG equations for the scalar case and describe how the usual loop expansion is obtained.

2. Massless scalar case

We illustrate in this case the main elements of the analysis. Some results are general and it is easier to illustrate them in the scalar case. We start by recalling the needed elements of Λ-RG and then we deduce in this formulation the beta function and compute the two-loop contribution. Then we introduce the improved perturbation expansion in terms of the flowing coupling and compare it with the usual running coupling. All these points will be reanalyzed in the Yang–Mills case in the next section.

2.1. Summary of Λ-RG formulation

We recall the Λ-RG formulation by taking as example the euclidean ϕ^4 massless scalar theory in four dimensions. One starts from the action at a UV scale Λ_0 larger than any physical scale in the theory

$$S_{\Lambda_0}[\phi] = \int d^4x \left\{ \frac{1}{2}\phi(x)[-Z^{\mathrm{UV}}\partial^2 + \Lambda_0^2 \rho_1^{\mathrm{UV}}]\phi(x) + \frac{\rho_2^{\mathrm{UV}}}{4!}\phi^4(x) \right\}. \tag{1}$$

All fields in the path integrals have frequencies smaller than Λ_0. Thus in Feynman diagrams all internal Euclidean momenta q_i are bounded by $q_i^2 \leqslant \Lambda_0^2$. The UV parameters ρ_i^{UV} are dimensionless. All other possible field monomials have coefficients which are suppressed by inverse powers of Λ_0 since they have negative dimension and here they are neglected for simplicity. Perturbatively this theory is renormalizable so that the UV parameters can be made dependent on Λ_0 in such a way that observables at the physical scales are independent of Λ_0 for $\Lambda_0 \to \infty$.

The dependence of the UV action on Λ_0 can be generalized to smaller values of the scale. Such a generalization, which is the basis for the Λ-RG formulation [1], is obtained by performing the path integrations over the fields with frequencies between Λ_0 and $\Lambda < \Lambda_0$. This gives the Wilsonian action $S_\Lambda[\phi]$ in which the coefficients of the various field monomials depend on Λ. Of course this action is not any more local.

An equivalent way to obtain the Wilsonian action follows from the observation that $S_\Lambda[\phi]$ generates Green functions in which the propagators have both the UV and an IR cutoff Λ,

$$D_{\Lambda\Lambda_0}(p) \equiv \frac{K_{\Lambda\Lambda_0}(p)}{p^2}\,, \tag{2}$$

with $K_{\Lambda\Lambda_0}(p) = 1$ in the region $\Lambda^2 \leqslant p^2 \leqslant \Lambda_0^2$ and rapidly vanishing outside. From this property one deduces [1,2] the RG evolution equation for the Wilsonian action,

$$\Lambda\partial_\Lambda e^{-S_\Lambda[\phi]} = \frac{1}{2}\int_q \Lambda\partial_\Lambda D_{\Lambda\Lambda_0}(q)\frac{\delta^2}{\delta\phi(q)\delta\phi(-q)}e^{-S_\Lambda[\phi]}\,, \qquad \int_q \equiv \int\frac{d^4q}{(2\pi)^4}\,. \tag{3}$$

For our study it is convenient to perform the Legendre transform of the Wilsonian action $S_\Lambda[\phi]$ and introduce the "cutoff effective action" $\Gamma[\phi;\Lambda,\Lambda_0]$ generating the "cutoff vertices" $\Gamma_{2n}(p_1,\ldots,p_{2n};\Lambda,\Lambda_0)$. The RG equations for these vertices, corresponding to (3), are schematically given in Fig. 1 and they are recalled with the proper details in Appendix A (see Refs. [7–9]). The RG equations (3) have to be supplemented by boundary conditions at some values of Λ. To discuss them one observes that at the UV scale $\Lambda = \Lambda_0$ the Wilsonian or cutoff effective action $\Gamma[\phi;\Lambda,\Lambda_0]$ reduces to the UV action (1)

$$\Gamma[\phi;\Lambda_0,\Lambda_0] = S_{\Lambda_0}[\phi]\,. \tag{4}$$

Moreover, when the IR and UV cutoff are removed ($\Lambda = 0$ and $\Lambda_0 \to \infty$) and the UV parameters ρ_i^{UV} have the proper dependence on Λ_0, the cutoff vertices $\Gamma_{2n}(\{p_i\};\Lambda,\Lambda_0)$ become the physical vertices $\Gamma_{2n}(\{p_i\})$

$$\Gamma_{2n}(\{p_i\}) = \Gamma_{2n}(\{p_i\};\Lambda = 0,\infty)\,. \tag{5}$$

The boundary conditions for the RG evolution equations are therefore given by the requirement of locality (4), i.e. by giving the values of the "relevant" UV parameters ρ_i^{UV} in (1) and by requiring that all parts of vertices with negative dimension vanish. Without

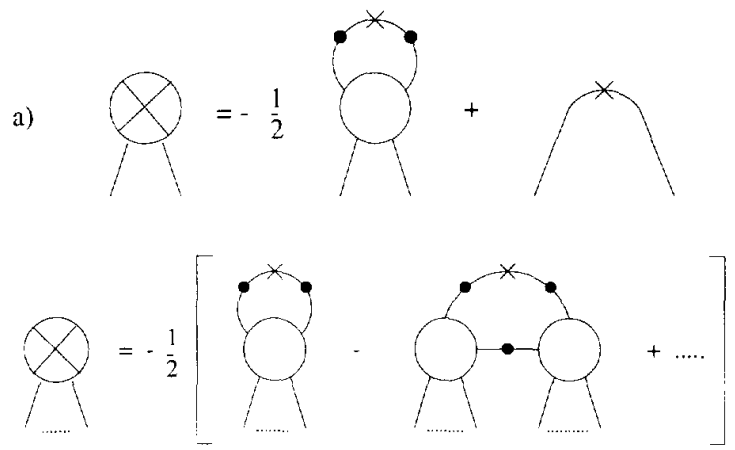

Fig. 1. Graphical representation of the Λ-RG equations for the two-point (a) and $2n$-point (b) vertex functions. The black dot represents the full propagator; open circles represent vertex functions with the IR cutoff Λ; crosses represent the derivative with respect to Λ of the free cutoff propagator and vertices. Integration over q in the loop is understood.

knowing the elementary theory at distances shorter than $1/\Lambda_0$, the UV parameters ρ_i^{UV} must be obtained from measurement at low scale. To define this procedure one has to introduce in general, for any Λ, the relevant parameters together with their irrelevant parts.

Relevant parameters, subtraction point and physical couplings. In the four-dimensional massless scalar ϕ^4 theory the two and four-point vertices Γ_2 and Γ_4 have non-negative mass dimensions and one has three relevant parameters defined, for instance, as follows

$$\Gamma_2(p;\Lambda,\Lambda_0) = D_{\Lambda\Lambda_0}^{-1}(p) + p^2(Z(\Lambda)-1) + \Lambda^2\rho_1(\Lambda) + \Delta_2(p;\Lambda,\Lambda_0)\,,$$
$$\Gamma_4(p_1,p_2,p_3,p_4;\Lambda,\Lambda_0) = \rho_2(\Lambda) + \Delta_4(p_1,p_2,p_3,p_4;\Lambda,\Lambda_0)\,, \tag{6}$$

with the conditions

$$\Delta_2(0;\Lambda,\Lambda_0) = \left.\frac{d\Delta_2(p;\Lambda,\Lambda_0)}{dp^2}\right|_{p^2=\mu^2} = 0\,, \qquad \Delta_4(\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{p}_4;\Lambda,\Lambda_0) = 0\,, \tag{7}$$

where μ is some subtraction scale and $\bar{p}_i$ is the symmetric point $\bar{p}_i\bar{p}_j = \mu^2(\delta_{ij} - \frac{1}{4})$. At the physical point, from (5) one has

$$Z(0) = 1\,, \qquad \Lambda^2\rho_1(\Lambda)|_{\Lambda=0} = 0\,, \qquad \rho_2(0) = g\,, \tag{8}$$

for large Λ_0. These conditions set the field normalization, the mass equal to zero and the coupling at μ.

The relevant parameters at the physical point $\Lambda = 0$ are related by the Λ-RG equations to the UV parameters ρ_i^{UV}. Therefore as boundary conditions one can take for the relevant parameters the physical values (8) and the vanishing of the irrelevant vertices at the UV point (locality condition (4)). In Appendix A we recall how the loop expansion can be generated from the RG equations for the cutoff vertices $\Gamma_{2n}(\{p_i\};\Lambda,\Lambda_0)$ with these boundary conditions.

For the relevant parameters $\rho_i(\Lambda)$ and $Z(\Lambda)$ we have explicitly written only the Λ dependence. However, from their definition in (6) they are dimensionless functions of Λ, Λ_0 and μ. One should write $\rho_i(\Lambda) \equiv \rho_i(g, \mu/\Lambda, \mu/\Lambda_0)$ and $Z(\Lambda) \equiv Z(g, \mu/\Lambda, \mu/\Lambda_0)$. In perturbation theory one can take the limit $\Lambda_0 \to \infty$ and all physical vertices become Λ_0 independent. This implies that one has for large Λ_0

$$\rho_i(\Lambda) \equiv \rho_i(g, \mu/\Lambda, \mu/\Lambda_0) = \rho_i(g, \mu/\Lambda, 0) + \mathcal{O}(\mu/\Lambda_0)\,, \tag{9}$$

and similarly for $Z(\Lambda)$. This allows one to simplify the perturbative analysis by putting $\Lambda_0 \to \infty$ from the beginning (see Ref. [7]). However, in the scalar ϕ^4 theory, non-perturbative triviality allows one to take this limit only for vanishing g.

2.2. Relation between Λ-RG and μ-RG

As is well known, renormalization is related to the UV property of the theory. One should then expect that μ-RG , which corresponds to the requirement that the physical observables do not depend on the specific value of μ, is related to Λ-RG for Λ in the UV region. Actually, as we will show, μ-RG corresponds to renormalizability, i.e. the independence of physical vertices on the UV cutoff Λ_0 for large Λ_0. The discussion on this point will be then in the framework of perturbation theory.

From (4) for Λ approaching the UV point Λ_0 the cutoff effective action $\Gamma[\phi; \Lambda, \Lambda_0]$ becomes the local UV action (1) and one can write

$$\begin{aligned}
\Gamma[\phi; \Lambda_0, \Lambda_0] &= S_{\Lambda_0}[\phi] \\
&= \int d^4x \left\{ \frac{1}{2}\phi^{\mathrm{UV}}(x)[-\partial^2 + \Lambda_0^2 \bar{\rho}_1^{\mathrm{UV}}]\phi^{\mathrm{UV}}(x) + \frac{\bar{\rho}_2^{\mathrm{UV}}}{4!}(\phi^{\mathrm{UV}}(x))^4 \right\}, \\
\phi^{\mathrm{UV}} &= \sqrt{Z^{\mathrm{UV}}}\phi\,, \qquad \bar{\rho}_1^{\mathrm{UV}} = \frac{\rho_1^{\mathrm{UV}}}{Z^{\mathrm{UV}}}\,, \qquad \bar{\rho}_2^{\mathrm{UV}} = \frac{\rho_2^{\mathrm{UV}}}{(Z^{\mathrm{UV}})^2}\,.
\end{aligned} \tag{10}$$

The UV parameters are given by

$$\rho_i^{\mathrm{UV}} \equiv \rho_i(g, \mu/\Lambda, \mu/\Lambda_0)|_{\Lambda=\Lambda_0}\,, \qquad Z^{\mathrm{UV}} = Z(g, \mu/\Lambda, \mu/\Lambda_0)|_{\Lambda=\Lambda_0}\,. \tag{11}$$

Consider now the physical vertices $\Gamma_{2n}(\{p_i\}, g, \mu)$ obtained from this UV action. The rescaling of the fields in (10) allows one to explicitly factorize the dependence on Z^{UV}

$$\Gamma_{2n}(\{p_i\}, g, \mu) = (Z^{\mathrm{UV}})^n \Gamma_{2n}^{\mathrm{UV}}(\{p_i\}, \bar{\rho}_1^{\mathrm{UV}}, \bar{\rho}_2^{\mathrm{UV}}; \Lambda_0)\,. \tag{12}$$

These UV vertices $\Gamma_{2n}^{\mathrm{UV}}$ are obtained from (10) by Feynman rules with internal propagators of momentum q with the UV cutoff $q^2 < \Lambda_0^2$. The various Feynman diagram contributions to $\Gamma_{2n}^{\mathrm{UV}}$ contain logarithmic divergent powers of

$$\ln\left(\frac{P^2}{\Lambda_0^2}\right) = \ln\left(\frac{P^2}{\mu^2}\right) + \ln\left(\frac{\mu^2}{\Lambda_0^2}\right)\,, \tag{13}$$

with P a combination of external momenta. The two-point vertex contains also quadratically divergent mass terms proportional to Λ_0^2. Renormalizability ensures that (perturbatively) all these Λ_0^2 divergences are cancelled by the divergent contributions in the

UV parameters $\rho_i^{\rm UV}$ and $Z^{\rm UV}$ so that for large Λ_0 the vertex functions are independent of Λ_0. This fact is expressed by the equation

$$\left[\Lambda_0\partial_{\Lambda_0} + (\Lambda_0\partial_{\Lambda_0}\bar\rho_1^{\rm UV})\frac{\partial}{\partial\bar\rho_1^{\rm UV}} + (\Lambda_0\partial_{\Lambda_0}\bar\rho_2^{\rm UV})\frac{\partial}{\partial\bar\rho_2^{\rm UV}} + n\Lambda_0\partial_{\Lambda_0}\ln Z^{\rm UV}\right] \times\Gamma_{2n}(\{p_i\},g,\mu) = 0\,, \tag{14}$$

valid for large $\Lambda_0 \to \infty$, i.e. the corrections are of order μ/Λ_0. Here ∂_{Λ_0} is the explicit derivative with respect to Λ_0 in (12).

To relate this equation to the μ-RG equation one needs to analyse the μ dependence of the vertices and then of the UV parameters. Since the field normalization does not affect physical observables, the two sets of vertices $\Gamma_{2n}(\{p_i\},g,\mu)$ with coupling g at a scale μ and $\Gamma_{2n}(\{p_i\},g',\mu')$ with coupling g' at a scale μ' correspond to the same theory if they are related by

$$\Gamma_{2n}(\{p_i\},g',\mu') = z^{-n}\Gamma_{2n}(\{p_i\},g,\mu)\,. \tag{15}$$

From (12), one has

$$z = z(g,\mu'/\mu) = Z^{\rm UV}/Z'^{\rm UV}\,,$$

with $Z'^{\rm UV}$ given in (11) as a function of g' and μ'. Then from (12) and (15) one deduces, for $\Lambda_0 \to \infty$, that the UV parameters $\bar\rho_1^{\rm UV}$ and $\bar\rho_2^{\rm UV}$ are independent of μ. From this property one obtains

$$\beta(g) \equiv \mu\partial_\mu g = \frac{\Lambda_0\partial_{\Lambda_0}\bar\rho_2^{\rm UV}}{\partial_g\bar\rho_2^{\rm UV}} = \frac{\Lambda_0\partial_{\Lambda_0}\bar\rho_1^{\rm UV}}{\partial_g\bar\rho_1^{\rm UV}}\,, \tag{16}$$

giving the relation between beta function and UV parameters we were searching for. Similarly, for the field anomalous dimension one has

$$\gamma_\phi(g) = -\tfrac{1}{2}\mu'\partial_{\mu'}\ln z(g,\mu'/\mu)|_{\mu'=\mu} = -\tfrac{1}{2}[\Lambda_0\partial_{\Lambda_0} - \beta(g)\partial_g]\ln Z^{\rm UV}\,. \tag{17}$$

Notice that the UV parameters are logarithmically divergent for $\Lambda_0 \to \infty$, but (perturbative) renormalizability ensures that, in this limit, $\beta(g)$ and $\gamma_\phi(g)$ are finite and dependent only on g.

It is easy to show that the renormalizability condition (14) is equivalent to the μ-RG equation. From (16) and (17) we can write (14) in the form ($\Lambda_0 \to \infty$)

$$\left[\Lambda_0\partial_{\Lambda_0} + \beta(g)\left(\partial_g\bar\rho_1^{\rm UV}\frac{\partial}{\partial\bar\rho_1^{\rm UV}} + \partial_g\bar\rho_2^{\rm UV}\frac{\partial}{\partial\bar\rho_2^{\rm UV}} + \partial_g Z^{\rm UV}\frac{\partial}{\partial Z^{\rm UV}}\right) - 2n\gamma_\phi(g)\right] \times\Gamma_{2n}(\{p_i\},g,\mu) = \left[\Lambda_0\partial_{\Lambda_0} + \beta(g)\partial_g - 2n\gamma_\phi(g)\right]\Gamma_{2n}(\{p_i\},g,\mu) = 0\,, \tag{18}$$

where the explicit Λ_0 dependence in the vertices is contained in $\Gamma_{2n}^{\rm UV}$ (see (12)). From (13) we see that the explicit logarithmic Λ_0 dependence in the UV vertices $\Gamma_{2n}^{\rm UV}$ is the same as the explicit μ dependence in the physical vertices Γ_{2n}. Then, $\Lambda_0\partial_{\Lambda_0}$ is equivalent to $\mu\partial_\mu$ and one recovers the usual μ-RG equation.

The fact that the theory is (perturbatively) renormalizable ensures that the Λ_0 dependence of the relevant parameters vanishes for $\Lambda_0 \to \infty$ (see (9)). This implies that we can introduce the relevant parameters $\rho_i(g, \mu/\Lambda, \mu/\Lambda_0)$ directly for $\Lambda_0 \to \infty$. From now on we shall use

$$\rho_i(\Lambda) = \rho_i(g, \mu/\Lambda, 0) , \qquad Z(\Lambda) = Z(g, \mu/\Lambda, 0) ,$$

so that the beta function in Eq. (16) can be expressed in terms of these functions for large Λ, i.e. $\mu/\Lambda \to 0$. For the beta function and the anomalous dimension one has [3]

$$\beta(g) = \left[\frac{\Lambda\partial_\Lambda \bar\rho_1}{\partial_g \bar\rho_1}\right]_{\mu/\Lambda=0} = \left[\frac{\Lambda\partial_\Lambda \bar\rho_2}{\partial_g \bar\rho_2}\right]_{\mu/\Lambda=0} , \tag{19}$$

$$\gamma_\phi(g) = -\tfrac{1}{2}[\Lambda\partial_\Lambda - \beta(g)\partial_g] \ln Z|_{\mu/\Lambda=0} ,$$

with $\bar\rho_2 = \rho_2/Z^2$ and $\bar\rho_1 = \rho_1/Z$. The parameters $\bar\rho_i$ have logarithmic divergent contributions for $\mu/\Lambda \to 0$ and only the ratios in (19) are finite. The relation (19) between the beta function and the parameters $\bar\rho_2$ was obtained in Ref. [5] to one-loop order where the denominator $\partial_g\bar\rho_2$ can be ignored. As we shall see at two loops the denominator is essential to obtain a finite result for $\mu/\Lambda \to 0$.

In the following we shall focus our attention on $\bar\rho_2$ which will be called the "Wilsonian flowing coupling", $\hat g(\Lambda) \equiv \bar\rho_2(\Lambda)$. We show now that for large Λ this coupling becomes equivalent to a running coupling at the momentum scale $k^2 = \Lambda^2 \gg \mu^2$. This is a consequence of the fact that for large Λ this coupling becomes independent of μ so that one can write

$$\hat g(\Lambda) \equiv \bar\rho_2(g, \mu/\Lambda) = G(g(\mu), \mu/\Lambda)(1 + \mathcal{O}(\mu/\Lambda)) ,$$

where the function G does not depend on μ. This means that, apart from power corrections, the μ dependence in $g(\mu)$ is completely cancelled by the explicit μ/Λ dependence. We then can write

$$\hat g(\Lambda) = G(g(\Lambda), 1)(1 + \mathcal{O}(\mu/\Lambda)) , \quad G(g(\Lambda), 1) = g(\Lambda) + \mathcal{O}(g^2(\Lambda)) , \tag{20}$$

so that $G(g(\Lambda), 1)$ itself is a possible running coupling at the scale Λ.

2.3. Beta function at two loops by Λ-RG

To illustrate in some details the Λ-RG formulation we compute the two-loop beta function and anomalous dimension using (19) with $\bar\rho_2$. We have to compute the behaviour for $\mu/\Lambda \to 0$ of the perturbative coefficients of ρ_2 and Z

[3] From the fact that the UV parameters $\bar\rho_i^{\rm UV}$ are independent of μ for large Λ_0 and from renormalizability we have $\bar\rho_i^{\rm UV} \equiv \bar\rho_i(g(\mu), \mu/\Lambda_0, \mu/\Lambda_0) = \bar\rho_i(g(\mu), \mu/\Lambda_0, 0) + \mathcal{O}(\mu/\Lambda_0)$. Then $\bar\rho_i(g(\mu), \mu/\Lambda, 0)$ are independent of μ for large Λ.

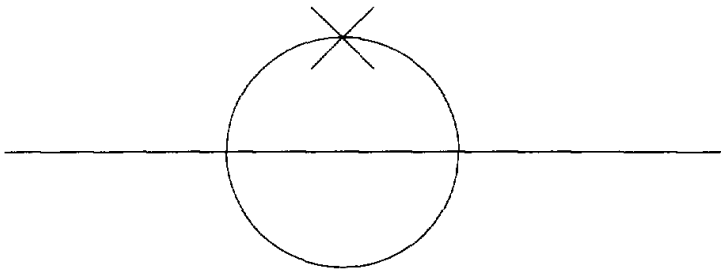

Fig. 2. Feynman diagram contributing to $\dot{Z}^{(2)}(\Lambda)$.

$$\rho_2(g,\mu/\Lambda) = g + \sum_{\ell=1}^{\infty} g^{\ell+1}\rho_2^{(\ell)}(\mu/\Lambda)\,,$$

$$Z(g,\mu/\Lambda) = 1 + \sum_{\ell=2}^{\infty} g^{\ell} Z^{(\ell)}(\mu/\Lambda)\,. \tag{21}$$

The corrections to Z start from two loops since the one-loop self-energy does not depend on the momentum. The first two coefficients of the beta function

$$\beta(g) = b_0 g^2 + b_1 g^3 + \ldots \tag{22}$$

are given by (see (19))

$$b_0 = \dot{\rho}_2^{(1)}|_{\mu=0}\,, \qquad b_1 = (\dot{\rho}_2^{(2)} - 2\dot{\rho}_2^{(1)}\rho_2^{(1)} - 2\dot{Z}^{(2)})|_{\mu=0}\,,$$
$$\dot{\rho}_i(\Lambda) \equiv \Lambda\partial_\Lambda \rho_i(\Lambda)\,. \tag{23}$$

As we shall see, in the expression for b_1 the divergence contribution in $\dot{\rho}_2^{(2)}$ is cancelled by the divergent contribution $-2\dot{\rho}_2^{(1)}\rho_2^{(1)}$, which is coming from $\partial_g\tilde{\rho}_2$ in the denominator of (19).

At one loop the relevant parameters are obtained from Fig. 1 with the vertices at tree level. The momentum dependence of the self-energy vanishes ($Z^{(1)} = 0$). Neglecting the subtraction point one finds

$$b_0 = \dot{\rho}_2^{(1)}|_{\mu=0} = -3\int_q \frac{\dot{K}_\Lambda(q)}{q^2}\frac{K_\Lambda(q)}{q^2} = -\frac{3}{2}\int_q \frac{1}{q^4}\Lambda\partial_\Lambda K_\Lambda^2(q) = 3r\,, \qquad r = \frac{1}{16\pi^2}\,. \tag{24}$$

Notice that since $\dot{\rho}_2^{(1)}(\mu/\Lambda)$ is a constant for $\mu/\Lambda = 0$, $\rho_2^{(1)}(\mu/\Lambda)$ diverges logarithmically as $\ln \mu/\Lambda$.

The two-loop coupling $Z^{(2)}(\mu/\Lambda)$ is obtained from Fig. 1a by taking the four-point function at one loop (to this order self-energy corrections in internal propagators do not contribute). Neglecting the subtraction point μ one has (see Fig. 2)

$$\dot{Z}^{(2)}|_{\mu=0} = -\frac{1}{6}\int_q\int_{q'} \Lambda\partial_\Lambda\partial_{p^2}\left\{\frac{K_\Lambda(q)K_\Lambda(q')K_\Lambda(q+q'+p)}{q^2q'^2(q+q'+p)^2}\right\}_{p=0} = -\frac{r^2}{6}\,. \tag{25}$$

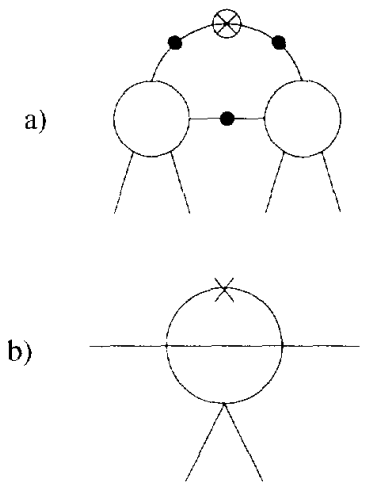

Fig. 3. (a) Graphical representation of the contribution to $\dot{\rho}_2$ (see the second term of (A.6)). (b) Feynman diagram contributing to $\dot{\rho}_2^{(2)}$ (see the first term of (A.6)).

These integrals are easily done by neglecting one of the cutoff function. The remaining contribution gives a constant which vanishes when the Λ-derivative is taken.

The two-loop coupling $\dot{\rho}_2^{(2)}(\mu/\Lambda)$ is obtained from the diagrams of Fig. 3a, in which the internal vertices are at one loop, and of Fig. 3b. The first contribution is

$$g^3\dot{\rho}_{2A}^{(2)}|_{\mu=0} = -\frac{3}{2}\int_q \left[\Gamma_4^2(q,0,0,-q;\Lambda)\Lambda\partial_\Lambda \frac{1}{\Gamma_2^2(q;\Lambda)}\right]_{\text{one loop}} . \tag{26}$$

We have to consider the three contributions $\rho_1^{(1)}(\Lambda)$, $\rho_2^{(1)}(\Lambda)$ and $\Delta_4^{(1)}(q,0,0,-q;\Lambda)$ coming from the one-loop corrections to the two- and four-point functions.

The contribution from $\rho_1^{(1)}(\Lambda)$ is zero. This is due to the fact that, before taking the Λ derivative, the integral is convergent, dimensionless and then its result is a constant in Λ.

The contribution from $\rho_2^{(1)}(\Lambda)$ is

$$\dot{\rho}_{2A'}^{(2)}|_{\mu=0} = -3\rho_2^{(1)}\int_q \Lambda\partial_\Lambda\left(\frac{K_\Lambda(q)}{q^2}\right)^2 = 2\rho_2^{(1)}\dot{\rho}_2^{(1)} . \tag{27}$$

As anticipated, this contribution is cancelled by the term $-2\rho_2^{(1)}\dot{\rho}_2^{(1)}$ in the combination (23) which is coming from the denominator $\partial_g\tilde{\rho}_2$.

The contribution from $\Delta_4^{(1)}(q,0,0,-q;\Lambda)$ is

$$\begin{aligned}\dot{\rho}_{2A''}^{(2)}|_{\mu=0} &= 3\int_q \Lambda\partial_\Lambda\left(\frac{K_\Lambda(q)}{q^2}\right)^2\int_{q'}\frac{K_\Lambda(q')}{q'^2}\left(\frac{K_\Lambda(q+q')}{(q+q')^2}-\frac{K_\Lambda(q')}{q'^2}\right)\\ &= 3\int_q \Lambda\partial_\Lambda\left(\frac{K_\Lambda(q)}{q^2}\right)^2\int_{q'}\frac{K_\Lambda(q')(K_\Lambda(q+q')-1)}{q'^2(q+q')^2} ,\end{aligned} \tag{28}$$

where the equality of the two integrals can be shown by performing the angular integration.

Finally, the contribution from the diagram of Fig. 3b is

$$\dot{\rho}^{(2)}_{2B}|_{\mu=0} = 3\int_{q'}\left(\frac{K_\Lambda(q')}{q'^2}\right)^2\int_q \Lambda\partial_\Lambda \frac{K_\Lambda(q)K_\Lambda(q+q')}{q^2(q+q')^2}\,.$$

It is easy to compute this integral if one of the cutoff functions inside the second integral is neglected

$$\dot{\rho}^{(2)}_{2B'}|_{\mu=0} = 3\int_{q'}\left(\frac{K_\Lambda(q')}{q'^2}\right)^2\int_q \Lambda\partial_\Lambda \frac{K_\Lambda(q)}{q^2(q+q')^2} = -6r^2\,. \tag{29}$$

The remaining term cancels exactly the contribution (28).

In conclusion the only non-cancelling contributions to the two loops beta function and anomalous dimension are given by (25) and (29) and one finds[4]

$$b_1 = -\frac{17}{3}r^2\,, \qquad \gamma^{(1)}_\phi(g) = \frac{1}{12}g^2r^2\,. \tag{30}$$

The evaluations given above are obtained essentially by using a theta function for the cutoff function, but the results do not depend on this specific assumption and depend essentially on dimensional counting of convergent integrals. This is due to the fact that one takes the limit $\Lambda \to \infty$ (a smooth representation of the theta function gives corrections vanishing as inverse powers of Λ).

2.4. *Flowing coupling and resummation*

As recalled in Section 1, it is usually assumed that in a Feynman diagram with a virtual loop of momentum k higher order corrections reconstruct the running coupling $g(k)$ at that momentum scale. We want here to show that the Λ-RG formulation may provide a simple framework in which one can study this resummation.

As we have seen in the previous subsections (see also Appendix A) the usual perturbative expansion is obtained by solving iteratively the Λ-RG evolution equations with the initial condition that at zero loop the only non-vanishing vertices are $\Gamma^{(0)}_4(p_1 \dots p_4; \Lambda) = g$ and $\Gamma^{(0)}_2(p;\Lambda) = D^{-1}_{\Lambda\Lambda_0}(p)$.

It is interesting to keep in mind how the usual Feynman diagrams are obtained by iterating the Λ-RG equations. This should be clear already after the two-loop beta function calculation of Subsection 2.3. In general one has the following structure. Suppose to know all vertices $\Gamma^{(\ell')}[\Lambda']$ with $\ell' < \ell$. Then (see Appendix A) the vertices $\Gamma^{(\ell)}[\Lambda]$ are obtained from $\Gamma^{(\ell')}[\Lambda']$ by adding a new loop of momentum k_ℓ with $k_\ell^2 = \Lambda'^2 \geqslant \Lambda^2$. Thus this additional momentum is larger than all loop momenta in the previous vertices.

[4] In Ref. [9] an approximate perturbative calculation of the beta function has been obtained using a momentum expansion and a truncation of the RG equations which omits the contribution of the six-point vertex.

Therefore by successive iteration one generates contributions of Feynman diagrams at ℓ loops in which the loop momenta k_i are ordered in virtuality $\Lambda^2 < k_1^2 < \ldots < k_\ell^2$. The complete Feynman diagrams at that loop are obtained by summing all orderings.

It is possible to formulate an improved iterative procedure in which one introduces the Wilsonian flowing couplings $\hat{g}(k_i)$ at the scale k_i of the successive momenta generated by the iteration. This improved perturbative expansion consists in solving the Λ-RG equations as for the usual perturbative expansion with the only difference that at zero loop the four-point function is given by the flowing coupling

$$\Gamma_4^{(0)}(p_1 \ldots p_4; \Lambda) = \hat{g}(\Lambda) \,. \tag{31}$$

The flowing coupling will be obtained later by a generalized beta function computed as an expansion in the Wilsonian coupling itself.

From (31) one starts the iteration by computing to the next loop the irrelevant part of the vertices and the two relevant parameters $Z(\Lambda)$ and $\rho_1(\Lambda)$ as function of the flowing coupling. After iterating this procedure ℓ times, the vertices are given in terms of flowing couplings $\hat{g}(k_i)$ at the scale k_i of the successive momenta generated by the iteration.

From the vertices obtained after ℓ iterations one computes the generalized beta function

$$\begin{aligned} B[\hat{g}; \Lambda] &\equiv \Lambda \partial_\Lambda \hat{g}(\Lambda) \\ &= -2\frac{\dot{Z}}{Z}\hat{g}(\Lambda) + \frac{1}{2}\int_k Z^{-2}(\Lambda)\bar{I}_6(k, \bar{p}_1 \ldots \bar{p}_4, -k; \Lambda)\Lambda\partial_\Lambda \frac{1}{\Gamma_2(k;\Lambda)} \,. \end{aligned} \tag{32}$$

The vertex $\bar{I}_6$ is defined in Appendix A in terms of the two-, four- and six-point cutoff vertices. The IR cutoff Λ is the virtuality $\sqrt{k^2}$ of the loop momentum in the successive iteration.

Solving (32) with the boundary condition $\hat{g}(0) = g$, one computes $\hat{g}(\Lambda)$ as a partial resummation of perturbative expansion. Notice that $B[\hat{g}; \Lambda]$ is a functional of the flowing coupling $\hat{g}(k_i)$ and the scale Λ. This has to be contrasted with $\beta(g)$ which is a function of g.

To illustrate the procedure we compute the flowing coupling to one-loop order (similar approximation has been considered for large Λ in Ref. [10]). This is obtained by taking the vertices in the integrand of (32) to zero loop

$$\bar{I}_6^{(0)}(k, p_1 \ldots p_4, -k; \Lambda) = -2\hat{g}^2(\Lambda)\left[D_{\Lambda\Lambda_0}(Q_{12}) + D_{\Lambda\Lambda_0}(Q_{13}) + D_{\Lambda\Lambda_0}(Q_{14})\right] \,, \tag{33}$$

where $Q_{ij} = k + p_i + p_j$. The equation for $Z(\Lambda)$ gives to this order $\dot{Z}^{(1)}(\Lambda) = 0$ and one obtains

$$B^{(1)}[\hat{g}; \Lambda] = b_0 \hat{g}^2(\Lambda) f(\mu/\Lambda) \,,$$

where b_0 is the one-loop coefficient of the beta function and

$$f(\mu/\Lambda) = -8\pi^2 \int_k \Lambda\partial_\Lambda \frac{K_\Lambda(k)\, K_\Lambda(k+\bar{p})}{k^2(k+\bar{p})^2} = 1 - \frac{2\mu}{3\pi\Lambda} + \mathcal{O}(\mu^2/\Lambda^2)\,, \quad \Lambda \to \infty\,. \tag{34}$$

For $\Lambda \to 0$ this function vanishes and one finds $f(\mu/\Lambda) = 2(\Lambda/\mu)^2 + \mathcal{O}(\Lambda/\mu)^3$.

To this order $\hat{g}(\mu/\Lambda)$ is obtained by solving

$$\Lambda\partial_\Lambda \hat{g}(\Lambda) = b_0\, \hat{g}^2(\Lambda) f(\mu/\Lambda)\,, \qquad \hat{g}(0) = g\,. \tag{35}$$

As expected, since $f(\mu/\Lambda) \to 1$ for large Λ, the flowing and running coupling become the same for large Λ

$$\hat{g}(\Lambda) = \frac{g}{1 - b_0 g \ln(\Lambda/\mu_0)} \left(1 + \mathcal{O}\left(\frac{\mu}{\Lambda}\right)\right), \tag{36}$$

where $\mu_0 \simeq \mu$. For smaller Λ the two couplings differ: $\hat{g}(0) = g$ while $g(\mu) = g$ and $g(k) \to 0$ for $k \to 0$.

The physical vertices are obtained by integrating the virtualities between $\Lambda = 0$ and Λ_0. The presence of the Landau pole at $k_L \simeq \mu\, e^{1/b_0 g}$ implies that the UV cutoff cannot be removed in this case, although the theory is perturbatively renormalizable. This corresponds to the property of triviality of the scalar theory, entailing that the limit $\Lambda_0 \to \infty$ is possible only if the coupling g vanishes.

3. Yang–Mills case

We show that the previous connection between μ-RG and Λ-RG formulations can be established also for Yang–Mills theory and we shall compute as an example the one-loop beta function. We recall that the Λ-RG for this theory is consistent with the BRS symmetry [11] provided the boundary conditions at the physical point $\Lambda = 0$ and $\Lambda \to \infty$ are properly chosen [4,6] (see also Ref. [12]). We then consider the iterative solution in terms of the Wilsonian couplings.

3.1. Summary of Λ-RG formulation

Consider as an example the $SU(2)$ Yang–Mills theory in which $\phi = (A^a_\mu, c^a, \bar{c}^a)$ are the vector and the ghost fields and $\gamma = (u^a_\mu, v^a)$ the sources for composite operators associated to the BRS variations of A^a_μ, and c^a respectively. The BRS classical action, in the Feynman gauge, is

$$S_{\mathrm{BRS}}[\phi, \gamma; g] = \int d^4x \left\{ -\frac{1}{4} F^2_{\mu\nu} - \frac{1}{2}(\partial_\mu A_\mu)^2 + W_\mu \cdot D_\mu c - \frac{1}{2}\, v \cdot c \wedge c \right\}, \tag{37}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \wedge A_\nu$, $D_\mu c = \partial_\mu c + g A_\mu \wedge c$, $W_\mu = \partial_\mu \bar{c} + u_\mu/g$ and we have introduced the usual scalar and external $SU(2)$ products.

In the Λ-RG formulation one introduces the cutoff effective action $\Gamma[\phi, \gamma; \Lambda, \Lambda_0]$ by using propagators with IR and UV cutoff at Λ and Λ_0 respectively and one deduces

the RG equation similar to (3). Then one has to fix the boundary conditions. First of all one assumes that at the UV point $\Lambda = \Lambda_0$ the cutoff effective action becomes local. Then one has to fix the relevant parameters of $\Gamma[\phi, \gamma; \Lambda, \Lambda_0]$, i.e. the coefficients of all independent fields monomials which have dimension not larger than four and are Lorentz and $SU(2)$ scalars. There are nine relevant parameters. Two are the wave function constants $Z_A(\Lambda)$ and $Z_c(\Lambda)$ for the vector and ghost fields respectively which are the coefficients of the monomials

$$\tfrac{1}{2}A_\mu(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\cdot A_\nu\,, \qquad W_\mu\cdot\partial_\mu c\,.$$

The remaining parameters $\rho_i(\Lambda)$ with $i = 1\ldots7$ can be taken as the coefficients of the following seven monomials

$$\tfrac{1}{2}\Lambda^2 A_\mu\cdot A_\mu\,, \qquad -\tfrac{1}{2}(\partial_\mu A_\mu)^2\,, \qquad (\partial_\nu A_\mu)\cdot A_\mu\wedge A_\nu\,,$$

$$W_\mu\cdot A_\mu\wedge c\,, \qquad \tfrac{1}{4}(A_\mu\wedge A_\nu)\cdot(A_\nu\wedge A_\mu)\,, \qquad \tfrac{1}{4}(A_\mu\cdot A_\mu)^2\,, \qquad -\tfrac{1}{2}v\cdot c\wedge c\,,$$

respectively. Notice that some of these monomials are not present in the classical action (37). As in the scalar case (see (6), (7)) these nine parameters are given in terms of vertex functions evaluated at the subtraction points at the scale μ.

At the physical point $\Lambda = 0$ and $\Lambda_0 \to \infty$ the effective action $\Gamma[\phi, \gamma; 0, \infty]$ satisfies the BRS symmetry. As shown in Refs. [4,6] this symmetry is ensured if one properly fixes the relevant parameters defined at a given subtraction point μ as boundary conditions at $\Lambda = 0$. Taking the relevant part of the effective action, i.e. the part which contains only the relevant parameters, one has

$$\Gamma_{\mathrm{rel}}[\phi, \gamma; 0, \infty] = S_{\mathrm{BRS}} + \hbar\int d^4x\left\{\frac{x_5}{4}(A_\mu\wedge A_\nu)\cdot(A_\nu\wedge A_\mu) + \frac{x_6}{4}(A_\mu\cdot A_\mu)^2 - \frac{1}{2}x_7\, v\cdot c\wedge c\right\}, \tag{38}$$

where the three parameters x_i are given in terms of irrelevant vertices evaluated at the subtraction points (see Ref. [6]). Therefore, as boundary conditions at the physical point $\Lambda = 0$ and $\Lambda_0 \to \infty$ one assumes

$$Z_A(0) = 1\,, \qquad Z_c(0) = 1\,,$$
$$\rho_1(0) = 0\,, \qquad \rho_2(0) = 1\,, \qquad \rho_3(0) = g\,, \qquad \rho_4(0) = g\,,$$
$$\rho_5(0) = g^2 + \hbar x_5\,, \qquad \rho_6(0) = \hbar x_6\,, \qquad \rho_7(0) = 1 + \hbar x_7\,.$$

With these conditions the Λ-RG equation can be solved by expanding in $\hbar$ and one obtains the usual perturbative expansion for the vertex functions which satisfy the Slavnov–Taylor identities.

3.2. Beta function in the Λ-RG formulation

As in the scalar case we consider the cutoff effective action at the UV point. For $\Lambda = \Lambda_0$ the functional $\Gamma[\phi, \gamma; \Lambda, \Lambda_0]$ becomes local and equal to the UV action $S_{\Lambda_0}[\phi, \gamma]$. As in (11) we introduce the UV wave function constants and UV couplings

$$Z_i^{\mathrm{UV}} = Z_i(g, \mu/\Lambda, \mu/\Lambda_0)|_{\Lambda=\Lambda_0}, \qquad \rho_i^{\mathrm{UV}} = \rho_i(g, \mu/\Lambda, \mu/\Lambda_0)|_{\Lambda=\Lambda_0}.$$

Rescaling the fields, the UV action is given by

$$\begin{aligned}\Gamma[\phi, \gamma; \Lambda_0, \Lambda_0] &= S_{\Lambda_0}[\phi, \gamma] \\ &= \int d^4x \Big\{ \tfrac{1}{2} A_\mu^{\mathrm{UV}} (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) \cdot A_\nu^{\mathrm{UV}} + W_\mu^{\mathrm{UV}} \cdot \partial_\mu c^{\mathrm{UV}} \\ &\quad - \tfrac{1}{2} v^{\mathrm{UV}} \cdot c^{\mathrm{UV}} \wedge c^{\mathrm{UV}} + \tfrac{1}{2}\Lambda_0^2 \bar\rho_1^{\mathrm{UV}} A_\mu^{\mathrm{UV}} \cdot A_\mu^{\mathrm{UV}} - \tfrac{1}{2}\bar\rho_2^{\mathrm{UV}} (\partial_\mu A_\mu^{\mathrm{UV}})^2 \\ &\quad + \bar\rho_3^{\mathrm{UV}} (\partial_\nu A_\mu^{\mathrm{UV}}) \cdot A_\mu^{\mathrm{UV}} \wedge A_\nu^{\mathrm{UV}} + \bar\rho_4^{\mathrm{UV}} W_\mu^{\mathrm{UV}} \cdot A_\mu^{\mathrm{UV}} \wedge c^{\mathrm{UV}} \\ &\quad + \frac{\bar\rho_5^{\mathrm{UV}}}{4} (A_\mu^{\mathrm{UV}} \wedge A_\nu^{\mathrm{UV}}) \cdot (A_\nu^{\mathrm{UV}} \wedge A_\mu^{\mathrm{UV}}) + \frac{\bar\rho_6^{\mathrm{UV}}}{4} (A_\mu^{\mathrm{UV}} \cdot A_\mu^{\mathrm{UV}})^2 \Big\},\end{aligned} \tag{39}$$

where

$$A_\mu^{\mathrm{UV}} = \sqrt{Z_A^{\mathrm{UV}}} A_\mu, \qquad W_\mu^{\mathrm{UV}} c^{\mathrm{UV}} = Z_c^{\mathrm{UV}} W_\mu c. \tag{40}$$

The UV parameters $\bar\rho_i^{\mathrm{UV}}$ are obtained as in (10) by the usual rescaling. For instance one has

$$\bar\rho_3^{\mathrm{UV}} = \rho_3^{\mathrm{UV}} (Z_A^{\mathrm{UV}})^{-3/2}, \quad \bar\rho_4^{\mathrm{UV}} = \rho_4^{\mathrm{UV}} (Z_c^{\mathrm{UV}})^{-1} (Z_A^{\mathrm{UV}})^{-1/2}, \quad \bar\rho_5^{\mathrm{UV}} = \rho_5^{\mathrm{UV}} (Z_A^{\mathrm{UV}})^{-2}.$$

The BRS source v^{UV} is rescaled with no other effect. In the following we consider only vertex functions involving the vector and ghost fields and neglect the v^{UV} source.

The beta function and the field anomalous dimensions are obtained as in the scalar case. After factorizing the UV wave functions (see (12)) the vertices depend only on the six UV parameters $\bar\rho_i^{\mathrm{UV}}$ with $i = 1, \ldots, 6$. Since the μ dependence is in the factorized wave function Z_i^{UV} one deduces that the six UV rescaled parameters $\bar\rho_i^{\mathrm{UV}}$ do not depend on the subtraction point. Then one finds the relation between the beta function and the UV parameters at $\Lambda_0 \to \infty$,

$$\beta(g) \equiv \frac{dg}{d\ln\mu} = \frac{\Lambda_0 \partial_{\Lambda_0} \bar\rho_i^{\mathrm{UV}}}{\partial_g \bar\rho_i^{\mathrm{UV}}}. \tag{41}$$

Similarly for the field anomalous dimensions one has

$$\gamma_i(g) = -\tfrac{1}{2}[\Lambda_0 \partial_{\Lambda_0} - \beta(g)\partial_g] \ln Z_i^{\mathrm{UV}}. \tag{42}$$

Due to renormalizability the relevant parameters $\bar\rho_i(g, \mu/\Lambda, \mu/\Lambda_0)$ and $Z_i(g, \mu/\Lambda, \mu/\Lambda_0)$ are finite for $\Lambda_0 \to \infty$. Moreover they diverge for $\Lambda/\mu \to \infty$. Therefore, as in the scalar

case, in (41) one can use the parameters $\bar{\rho}_i(g, \mu/\Lambda, 0)$ and $Z_i(g, \mu/\Lambda, 0)$ and then take the limit $\mu/\Lambda \to 0$.

The one-loop expression for all the nine parameters has been obtained in Ref. [6]

$$\dot{Z}_A^{(1)}|_{\mu=0} = \frac{20}{3}r\,, \qquad \dot{Z}_c^{(1)}|_{\mu=0} = 2r\,, \qquad r \equiv \frac{1}{16\pi^2}\,,$$
$$\dot{\rho}_1^{(1)}|_{\mu=0} = \dot{\rho}_2^{(1)}|_{\mu=0} = \dot{\rho}_6^{(1)}|_{\mu=0} = 0\,,$$
$$\dot{\rho}_3^{(1)}|_{\mu=0} = \frac{8}{3}r\,, \qquad \dot{\rho}_4^{(1)}|_{\mu=0} = -2r\,, \qquad \dot{\rho}_5^{(1)}|_{\mu=0} = -\frac{4}{3}r\,. \tag{43}$$

According to (41) one finds the well-known one-loop beta function ($\beta(g) = b_0 g^3 + \ldots$) in the following three ways:

$$b_0 = (\dot{\rho}_3^{(1)} - \frac{3}{2}\dot{Z}_A)^{(1)}|_{\mu=0} = (\dot{\rho}_4^{(1)} - \frac{1}{2}\dot{Z}_A^{(1)} - \dot{Z}_c^{(1)})|_{\mu=0}$$
$$= \frac{1}{2}(\dot{\rho}_5^{(1)} - 2\dot{Z}_A^{(1)})|_{\mu=0} = -\frac{22}{3}r\,. \tag{44}$$

The one-loop beta function cannot be obtained from $\dot{\rho}_1^{(1)}$, $\dot{\rho}_2^{(1)}$ and $\dot{\rho}_6^{(1)}$ since they vanish to this order.

3.3. *Flowing couplings and resummation*

We extend to the Yang–Mills case the improved perturbation discussed in the scalar case. Here we limit our discussion to the one-loop improved expansion. To this end we have to give the effective action to zero loop. This is provided by the classical BRS action in (37) with the coupling g replaced by the Wilsonian coupling $\hat{g}(\Lambda)$ (all fields have frequencies between Λ and Λ_0)

$$\Gamma^{(0)}[\phi, \gamma; \Lambda] = S_{\mathrm{BRS}}[\phi, \gamma; \hat{g}(\Lambda)] \tag{45}$$

and boundary condition $\hat{g}(0) = g$. As in the scalar case, to obtain the generalized beta function, we have to identify $\hat{g}(\Lambda)$ with one of the relevant parameters $\bar{\rho}_i(\Lambda)$. For instance we take $\hat{g}(\Lambda) = \bar{\rho}_4(\Lambda)$, i.e. $\hat{g}(\Lambda)$ is given by the vector-ghost relevant coupling.

By iterating the Λ-RG equation once one obtains

$$\frac{\dot{Z}_A}{Z_A} = \left(\frac{19}{3}f_1(\mu/\Lambda) + \frac{1}{3}f_2(\mu/\Lambda)\right) r\hat{g}^2(\Lambda)\,,$$
$$\frac{\dot{Z}_c}{Z_c} = 2r f_3(\mu/\Lambda)\hat{g}^2(\Lambda)\,, \tag{46}$$

where the functions $f_i(\mu/\Lambda)$ take into account the presence of the cutoff function in the various one-loop Feynman graphs. They are similar to the function $f(\mu/\Lambda)$ introduced in the scalar case in (34). The one-loop generalized beta function is obtained by studying the ghost-vector vertex and one finds

$$\Lambda\partial_\Lambda \hat{g}(\Lambda) = -\left(\frac{1}{2}\frac{\dot{Z}_A}{Z_A} + \frac{\dot{Z}_c}{Z_c}\right)\hat{g}(\Lambda) - \left(\frac{1}{2}f_3(\mu/\Lambda) + \frac{3}{2}f_4(\mu/\Lambda)\right) r\hat{g}^3(\Lambda)\,, \tag{47}$$

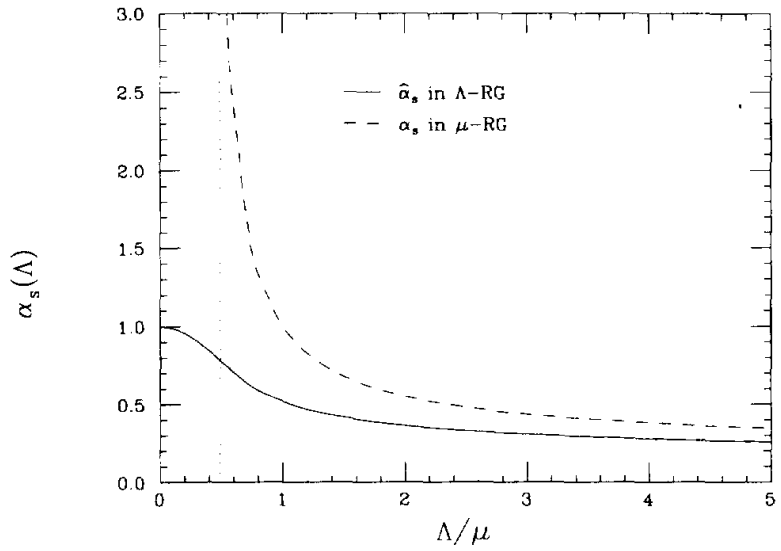

Fig. 4. Plot of the one-loop running coupling $\alpha_s(\Lambda)$ and the general pattern of the flowing coupling $\hat{\alpha}_s(\Lambda)$ in Yang–Mills theory as function of Λ/μ. We use $\alpha_s(\Lambda) \equiv g^2(\Lambda)/4\pi$ and $\hat{\alpha}_s(\Lambda) \equiv \hat{g}^2(\Lambda)/4\pi$ with $\alpha_s(\mu) = \hat{\alpha}_s(0) = 1$.

where again the two functions f_3 and f_4 take into account the cutoff. All functions f_i vanish quadratically for $\Lambda \to 0$ and they tend to one for $\mu/\Lambda \to 0$, apart for power corrections. Therefore one finds that for large Λ the flowing coupling tends to the one-loop running coupling

$$\hat{g}(\Lambda) \simeq g(\Lambda)\,, \tag{48}$$

with

$$g^2(k) = \frac{g^2}{1 - b_0 g^2 \ln(k^2/\mu^2)}\,, \qquad b_0 = -\frac{22}{3}r\,. \tag{49}$$

As an illustration we consider the case in which all functions $f_i(\mu/\Lambda)$ to be included in (46) and (47) are the same and given by the one obtained in the scalar case (34). Then the flowing coupling satisfies the RG equation

$$\Lambda\partial_\Lambda \hat{g}(\Lambda) = b_0 \hat{g}^3(\Lambda) f(\mu/\Lambda)\,, \qquad \hat{g}(0) = g\,. \tag{50}$$

In Fig. 4 we plot the solution of this equation together with the corresponding running coupling $g^2(\Lambda)$ in (49). For large Λ we have $\hat{g}(\Lambda) \simeq g(\Lambda)$. By decreasing the scale the power corrections in $f(\mu/\Lambda)$ start to become important and the two couplings start to differ. The one-loop running coupling has a Landau ghost at $k_L^2 = \mu^2 e^{1/b_0 g^2} < \mu^2$. The flowing coupling $\hat{g}(\Lambda)$ instead remains finite in all range of Λ and reaches the physical value $g = g(\mu)$ at $\Lambda = 0$.

As a result of this first improvement one has that the physical vertices are obtained by integrating $\hat{g}(k)$ for $k^2 > 0$. Recall that the IR cutoff Λ is the virtuality $\sqrt{k^2}$ of the loop momentum in the successive iteration. Since $\hat{g}(k)$ has no IR Landau pole one can compute the physical vertices without problems in the IR integration region.

4. Conclusion

We have studied the relation between μ-RG and Λ-RG , the formulations of renormalization group based on the two different ways of introducing the momentum scale. While in μ-RG the fundamental quantity is the running coupling $g(k)$ (fixed to be g at $k^2 = \mu^2$), in Λ-RG the natural couplings are the Wilsonian flowing couplings $\hat{g}_i(\Lambda)$ (fixed to be g at $\Lambda = 0$).

At large scale $\Lambda = k$ the two couplings differ only by power corrections of μ/Λ. This implies that the characteristic quantities of μ-RG formulation are obtained in the Λ-RG in the UV region of Λ. In particular the beta function is given by (19) and (41) and is obtained by using the fact that $\hat{g}(\Lambda)$ does not depend on μ for $\mu/\Lambda \to 0$. Moreover the Callan–Symanzik equation, i.e. independence of μ of the physical quantities, corresponds to the renormalizability condition, i.e. independence of the UV cutoff Λ_0 of physical quantities for $\Lambda_0 \to \infty$.

We have shown that it is possible to set up an iterative procedure to solve the Λ-RG equation in such a way that one generates Feynman diagram contributions in which the loop momenta are ordered ($\Lambda^2 < k_1^2 \ldots < k_\ell^2$) and the flowing couplings at these momenta are involved. As long as Λ is large, one can approximate the flowing couplings in terms of the running coupling. This justifies the usual assumption that in Feynman diagrams with a virtual momentum k higher order corrections generate the running coupling at that scale. This fact suggests that the Λ-RG formulation could provide a systematic method to study the problem of IR renormalons in non-Abelian gauge theory [13].

As Λ decreases and reaches the physical point $\Lambda = 0$ one has to take into account that the flowing couplings differ from the running coupling at low frequencies. At low frequencies, where the perturbative running coupling has a Landau ghost, the flowing couplings are finite all the way down to zero frequence. This allows then to perform the loop integration without problem in the IR region.

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We would like to thank C. Destri for helpful discussions.

Appendix A

We recall the Λ-RG equations for the vertex functions in the ϕ^4 massless scalar theory in the form used in the text. The cutoff vertex functions $\Gamma_{2n}(p_1 \ldots p_{2n}; \Lambda, \Lambda_0)$ have internal propagators

$$D_{\Lambda\Lambda_0}(q) = \frac{1}{q^2} K_{\Lambda\Lambda_0}(q) , \tag{A.1}$$

with $K_{\Lambda\Lambda_0}(q) = 1$ for $\Lambda^2 \leqslant q^2 \leqslant \Lambda_0$ and rapidly vanishing outside this region.

From the definition of the cutoff vertices and the fact that the Λ-dependence is only inside the cutoff propagator (A.1) one deduces [7–9] the following Λ-RG equations (see Fig. 1) (we omit to write the Λ_0 dependence in the vertex functions)

$$\dot{\Gamma}_2(p;\Lambda) = \dot{D}^{-1}_{\Lambda\Lambda_0}(p) - \frac{1}{2}\int_q \Gamma_4(q,p,-p,-q;\Lambda)\frac{1}{\Gamma_2^2(q;\Lambda)}\dot{D}^{-1}_{\Lambda\Lambda_0}(q)\,, \tag{A.2}$$

$$\dot{\Gamma}_{2n}(p_1\ldots p_{2n};\Lambda) = -\frac{1}{2}\int_q \bar{\Gamma}_{2n+2}(q,p_1\ldots p_{2n},-q;\Lambda)\frac{1}{\Gamma_2^2(q;\Lambda)}\dot{D}^{-1}_{\Lambda\Lambda_0}(q)\,, \tag{A.3}$$

where the associated vertices $\bar{\Gamma}_{2n+2}$ are one-particle reducible with respect to combinations of momenta including either q or $-q$ and one-particle irreducible with respect to all other combinations of momenta. For instance for $n=3$ one has (we omit the Λ and Λ_0 dependence)

$$\begin{aligned}\bar{\Gamma}_6(q,p_1\ldots p_4,-q) &= \Gamma_6(q,p_1\ldots p_4,-q)\\ &\quad -\sum \Gamma_4(q,p_1,p_2,Q_{12})\frac{1}{\Gamma_2(Q_{12})}\Gamma_4(-Q_{12},p_3,p_4,-q)\end{aligned} \tag{A.4}$$

with $Q_{ij}=q+p_i+p_j$ and the sum is over the six permutations of p_1,p_2,p_3,p_4. A simpler form for the Λ-RG equations is obtained by introducing the kernel $I_{2n+2}(q,p_1\ldots p_{2n},-q)$ which is one-particle irreducible and also two-particle irreducible with respect to the two momenta q and $-q$. We have then

$$\begin{aligned}\Gamma_{2n+2}(q\ldots -q) &= I_{2n+2}(q\ldots -q)\\ &\quad -\frac{1}{2}\int_{q'}\bar{I}_{2n+2}(q'\ldots -q')\frac{1}{\Gamma_2^2(q';\Lambda)}\Gamma_4(q,q',-q',-q)\,,\end{aligned} \tag{A.5}$$

where, for instance for $n=3$,

$$\begin{aligned}\bar{I}_6(q,p_1\ldots p_4,-q) &= I_6(q,p_1\ldots p_4,-q)\\ &\quad -\sum \Gamma_4(q,p_1,p_2,Q_{12})\frac{1}{\Gamma_2(Q_{12})}\Gamma_4(-Q_{12},p_3,p_4,-q)\,,\end{aligned} \tag{A.6}$$

with the sum over the permutations.

From this equation and by using (A.2) we obtain

$$\dot{\Gamma}_2(p;\Lambda) = \dot{D}^{-1}_{\Lambda\Lambda_0}(p) + \frac{1}{2}\int_q I_4(q,p,-p,-q;\Lambda)\Lambda\partial_\Lambda\frac{1}{\Gamma_2(q;\Lambda)}\,, \tag{A.7}$$

$$\dot{\Gamma}_{2n}(p_1\ldots p_{2n};\Lambda) = \frac{1}{2}\int_q \bar{I}_{2n+2}(q,p_1\ldots p_{2n},-q;\Lambda)\Lambda\partial_\Lambda\frac{1}{\Gamma_2(q;\Lambda)}\,. \tag{A.8}$$

In the text we discuss in particular the equations for the relevant parameters $\rho_1(\Lambda)$, $\rho_2(\Lambda)$ and $Z(\Lambda)$

$$\Lambda\partial_\Lambda(\Lambda^2\rho_1) = \frac{1}{2}\int_q I_4(q,0,0,-q;\Lambda)\Lambda\partial_\Lambda\frac{1}{\Gamma_2(q;\Lambda)},$$

$$\dot{Z} = \frac{1}{2}\int_q \partial_{\bar{p}^2} I_4(q,\bar{p},-\bar{p},-q;\Lambda)\Lambda\partial_\Lambda\frac{1}{\Gamma_2(q;\Lambda)}, \tag{A.9}$$

$$\dot{\rho}_2 = \frac{1}{2}\int_q \bar{I}_6(q,\bar{p}_1\ldots\bar{p}_4,-q;\Lambda)\Lambda\partial_\Lambda\frac{1}{\Gamma_2(q;\Lambda)}, \tag{A.10}$$

with $\bar{p}_i$ the subtraction point at the scale μ.

The usual loop expansion is obtained by solving iteratively these equations with the initial condition that at zero loop the vertices are given by the two-point function $\Gamma_2^{(0)}(p;\Lambda) = D^{-1}_{\Lambda\Lambda_0}(p)$ and the four-point function $\Gamma_4^{(0)}(p_1\ldots p_4;\Lambda) = g$.

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