Coherent bubble-sum approximation for coupled-channel resonance scattering

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For coupled-channel resonance scattering we derive a model with a closed form solution for the T-matrix that satisfies unitarity and analyticity. The two-channel case is handled explicitly for an arbitrary number of resonances. The method focuses on the expansion of the transition matrix elements, $\Gamma(s)$, in known analytical functions. The appropriate hadronic form factors and the related energy shifts can be determined from the scattering data. The differences between this method and the K-matrix and the Breit-Wigner approximation are illustrated in the case of the S_{11} resonances $S_{11}(1535)$ and $S_{11}(1650)$.

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I. INTRODUCTION

Analyticity is numerically the hardest problem to keep track of in a scattering problem. Both the singularities from on-shell states, which yield imaginary parts, and the effects of thresholds require careful analysis. This is one of the main hurdles for the development of unitary models for coupled-channel scattering.

In this short paper a set of consistent approximations that yield unitary and analytical results for coupled channel problems is described. It is accompanied by the open source program BUBBLEGUM, which yield numerical T-matrix results, together with the corresponding K-matrix approximation, [1] Breit-Wigner approximation, [2, 3] and the perturbative results. The general purpose of both the paper and the computer program is threefold. First, to show that approximations that violate unitarity or analyticity will generally deviate considerably [4] from exact result in the case of strong interactions and thresholds close to resonance values. Second, to show that the exact results can be derived straightforwardly with a reasonable effort, in a manner that can improved systematically. Third, to show that scattering data do not only contain pole information of the resonances but also the non-perturbative matrix elements between asymptotic, or scattering states, and resonances. Extraction of these matrix elements from the scattering data might lead to better constraints on models for hadrons.

Furthermore, substantial energy shifts occur through coupling of resonances with asymptotic states, or decay channels, which can only be analyzed in a fully analytic approach. Since hadrons are composite particles, they have form factors that suppress high-energy processes. These form factors and the energy shifts are related and should be treated consistently, which neither the K-matrix approach nor the Breit-Wigner approximation can do. Depending on the large energy behavior of form factor, the shift can vary substantially. In the case of suppression at high energy, the shift will change sign as scattering energy increases, from negative to positive.

It is common folkore that bare masses have no meaning. It stems from renormalizable field theory, where indeed masses can undergo abritrary shifts through finite renormalization. However, in hadron theory constituent quark models and other hadronic models yield the bare states with the bare masses and their couplings to the decay channels. In this case a comparison between scattering data and these models should incorporate the energy shifts through the non-perturbative coupling of bare states and decay channels.

II. THEORY

The simplest process contributing to s-channel scattering is the tree diagram, where the two-particle state j, in a particular partial wave, forms a resonance state b which decays into a two-particle state i. The transition amplitude is given by:

$$T_{ij}^{(0)} = g_{ib} f_i(E) \sqrt{\Gamma_i(E)} \frac{1}{E - M_b + i\epsilon} \sqrt{\Gamma_j(E)} f_j(E) g_{jb}^* \quad , \tag{1}$$

where g_{ib} and g_{jb} are the coupling constants, $\Gamma(E)$ is the phase space for each state, $f(E)$ the form factors, M_b the mass of the state b, and E the scattering energy.

This lowest order diagram is an poor approximation [5] to the actual scattering process, especially for hadron dynamics. First, the coupling constant is large, therefore, during scattering, the initial state j goes, on average, through several intermediate states, before ending up in the final state i. Second, for scattering energies E close to the resonance mass M, the interaction is strong due to the small energy denominators, even if the coupling is small. Hence, the simple tree diagram, which corresponds to the lowest order perturbative expansion, fails here. If the problem would be solved consistently, the resonance will acquire a width, which corresponds to the shift of the pole into the complex plane.
Solving the scattering problem consistently means iterating the elementary scattering processes such that it reaches a fixed point. There are many more or less equivalent ways to find the self-consistent solution. In this paper we will use the Lipmann-Schwinger equation [2, 6]:

$$T_{ca} = T^{(0)}_{ca} + \sum_d T^{(0)}_{cd} \frac{1}{E - E_d + i\epsilon} T_{da} \quad , \tag{2}$$

which can be written down in many other forms. In practice, this equation determines the transition amplitude T, which is the sum over all possible processes, with different numbers of intermediate states.
For the case of two coupled channels the matrix Lipmann-Schwinger equation will look like:

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = X + X \cdot \begin{pmatrix} iW_1 + \tilde{W}_1 & 0 \\ 0 & iW_2 + \tilde{W}_2 \end{pmatrix} \cdot \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \quad , \tag{3}$$

where

$$X = \left(\sum_a \frac{1}{E - M_a} \begin{pmatrix} |g_{1a}|^2 & g_{1a} g^*_{2a} \\ g^*_{1a} g_{2a} & |g_{2a}|^2 \end{pmatrix} \right) \quad . \tag{4}$$

The reduced transition amplitudes t are defined as:

$$T_{ij}(E) = |i\rangle t_{ij}(E) \langle j| \quad , \tag{5}$$

with the states $|i\rangle$ given by:

$$|i\rangle = \int dE \; \sqrt{\Gamma_i(E)} f_i(E) |\phi_i(E)\rangle \quad . \tag{6}$$

$|\phi_i(E)\rangle$ is the asymptotic state i with energy E.
The solution for this case of two coupled channels can still be given in a simple closed form:

$$t_{11} = \frac{X_{11} - \det[\mathrm{X}] \mathcal{W}_2}{1 - X_{11}\mathcal{W}_1 - X_{22}\mathcal{W}_2 + \det[\mathrm{X}]\mathcal{W}_1\mathcal{W}_2} \quad , \tag{7}$$

$$t_{22} = \frac{X_{22} - \det[\mathrm{X}] \mathcal{W}_1}{1 - X_{11}\mathcal{W}_1 - X_{22}\mathcal{W}_2 + \det[\mathrm{X}]\mathcal{W}_1\mathcal{W}_2} \quad , \tag{8}$$

$$t_{12} = \frac{X_{12}}{1 - X_{11}\mathcal{W}_1 - X_{22}\mathcal{W}_2 + \det[\mathrm{X}]\mathcal{W}_1\mathcal{W}_2} \quad , \tag{9}$$

$$t_{21} = \frac{X_{21}}{1 - X_{11}\mathcal{W}_1 - X_{22}\mathcal{W}_2 + \det[\mathrm{X}]\mathcal{W}_1\mathcal{W}_2} \quad , \tag{10}$$

where $\mathcal{W}_j$ is defined as: $\mathcal{W}_j = iW_j(E) + \tilde{W}_j(E)$, and the functions W are defined through the following integrals:

$$W_i(E) = \pi \int dE' \langle \phi_i(E') | \phi_i(E) \rangle \Gamma_i(E) f_i(E)^2 = \pi \Gamma_i(E) f_i(E)^2 \quad , \tag{11}$$

$$\tilde{W}_i(E) = \mathrm{PV} \int dE' \frac{\Gamma_i(E') f_i(E')^2}{E - E'} \quad , \tag{12}$$

where PV stands for a principal value integral, and the asymptotic states are orthonormal: $\langle \phi_i(E') | \phi_i(E) \rangle = \delta(E' - E)$. It is clear that these results can easily be extended to incorporate an arbitrary number of channels.

Both Γ and f are real; any phase information is to be incorporated in the coupling constants. In the absence of a form factor f the second integral generally does not converge. It requires regularization, and possibly a motivation for this regularization through renormalization. Subtracted dispersion integrals effectively correspond to regularization through local subtractions. Another option is to set $\tilde{W}$ to zero, and only use the phase space for the imaginary part, which is known as the K-matrix method.
For a general case the dispersion integral Eq. (12) cannot be solved analytically. However, since the form factor is rather arbitrary, it can be chosen such that $\Gamma_i(E)f_i(E)^2$ has a known solution for the dispersion integral. An expansion in a set of these functions, allows one to determine $\Gamma(E)f(E)^2$ as part of the analysis of scattering data of resonances. This quantity can, and should, be compared to models of hadrons since the function $\Gamma(E)|gf(E)|^2$ is the square of the transition matrix element of the Hamiltonian:

$$\Gamma(E)|gf(E)|^2 = |\langle\phi(E)|H|M\rangle|^2 \quad , \tag{13}$$

where M is the bare resonance state. The transition matrix element is expected to fall-off at higher energies, due to the composite nature of hadrons.
Another constraint on $W(E)$ is given by general covariance, which requires that $W(E)$ is analytic in $s = E^2$ with all the cuts on the real axis associated with decay channels, or asymptotic states. In Feynman perturbation theory covariance is the result of summing the different time-ordered diagrams. In the language of states, these different time orderings correspond to different states, which often correspond to decay channels that will be of little significance in the kinematical region of interest. However, as restoring covariance comes at little cost, we replace the energy dispersion relation with:

$$\tilde{W}(E') = \int dE \frac{W(E)}{E'-E} \quad \rightarrow \quad \int ds \frac{W(\sqrt{s})}{E'^2 - s} \quad , \tag{14}$$

would yield generally a small difference in $\tilde{W}$. This, however, includes the second time ordering and restores analyticity in s, which is a consequence of full covariance. Moreover, analyticity in s implies a restriction on possible approximations for $\tilde{W}(E)$, *i.e.*, there should not be a cut at $s = 0$. In $W(E)$ itself the second time ordering is not included, as it would correspond to a different decay state which is generally of little consequence, however, would yield additional singularities to keep track of. This feature of manifest relativistic invariant formulation makes the evaluation of higher order Feynman diagrams in Minkowski space complicated. The advantage of one-state-one-singularity approach advocated here is that the singular structure of the perturbative kernel of the Lipmann-Schwinger equation $T^{(0)}$ is near to trivial. Note, however, as we restrict the kinematical domain of validity to the energies where only the designated states, that appear as channels in the Lipmann-Schwinger equation, can go on shell, the result is fully covariant with the replacement Eq. (14).
Given a particular threshold behavior n, associated with a partial wave, we use as leading order approximation for $W(E)$:

$$W^{(0)}(\sqrt{s}) = \sqrt{s_{\rm th}} \frac{(s/s_{\rm th} - 1)^{n/2}}{(s/s_{\rm th})^{[n/2]+1}} \theta(s/s_{\rm th} - 1) \quad , \tag{15}$$

where $s_{\rm th}$ is the threshold energy squared, and $[i]$ is the integer part of i. The power n is not only related to the partial wave, but also to kinematical factors which are different for a heavy-light system like πN and an equal mass system, like $\pi\pi$. Furthermore, it depends as well on the kinematical range of the threshold, *i.e.*, if the threshold extends to where the masses are comparable to the energy, it would yield a different threshold behavior. For example, in the πN scattering in the Δ region, the real threshold behavior stops at about 50 MeV above the threshold.
An additional expansion to model or fit the function W away from threshold is given by a polynomial P_m of order m in $\xi = (s - s_{\rm th})/s$:

$$W(\sqrt{s}) = \theta(s/s_{\rm th} - 1) \frac{s_{\rm th}^{[n/2]-n/2+3/2}}{s^{[n/2]-n/2+1}} \xi^{n/2} P_m(\xi) \tag{16}$$

$$= \theta(s/s_{\rm th} - 1)\sqrt{s_{\rm th}} \sum_{i=0}^{m} c_i \frac{(s/s_{\rm th} - 1)^{n/2+i}}{(s/s_{\rm th})^{[n/2]+i+1}} \tag{17}$$

$$\equiv \theta(s/s_{\rm th} - 1)\sqrt{s_{\rm th}} \sum_{i=0}^{m} c_i w_{(n/2+i)([n/2]+i+1)} \quad , \tag{18}$$

such that $\tilde{W}$ is analytical in $s = 0$, and has the dimension of energy. The first coefficient $c_0 = 1$ such that the threshold behavior is determined solely by the power n and the coupling constant. The coupling constants g_{ib} are dimensionless. Note, that no additional dimensionful quantities are introduced beside the threshold energy, which makes a good candidate for a normalized description of resonance scattering, which is up to now littered with different form factors with all kind of dimensionful quantities with little meaning.
Each of these terms in the polynomial expansion has a closed form solution for the dispersion integral, where we distinquish integer and half-integer values for n in w_{nk}:

$$
\begin{aligned}
w_{nk} &= \theta(\hat{s}-1)\pi\frac{(\hat{s}-1)^n}{\hat{s}^k} && (19)\\
\tilde{w}_{nk} &= (1-\mathcal{T}_0)\frac{(\hat{s}-1)^n \log|1-\hat{s}|}{\hat{s}^k} && (20)\\
w_{\frac{2n+1}{2}k} &= \pi\frac{\mathrm{Re}(\hat{s}-1)^{(2n+1)/2}}{\hat{s}^k} && (21)\\
\tilde{w}_{\frac{2n+1}{2}k} &= (-1)^n(1-\mathcal{T}_0)\pi\frac{\mathrm{Re}(1-\hat{s})^{(2n+1)/2}}{\hat{s}^k} && (22)
\end{aligned}
$$

where $\mathcal{T}_0$ refers to a Taylor-Laurent expansion of all the singular terms in s^{-1}, and $\hat{s} = s/s_{\mathrm{th}}$. The function $\xi^m/\hat{s}$ Eq. (16) peaks at $s = (m+1)s_{\mathrm{th}}$, while $\xi^m/\sqrt{\hat{s}}$ peaks at $s = (2m+1)s_{\mathrm{th}}$, such that every higher order term in $P_m(\xi)$ extend the range of energies over which $W(E)$ can be fitted. In a mathematical sense, the polynomials form a L^1 basis for W on $0 < 1/(1+\xi) < 1$.
The real part $\tilde{w}$ can be derived in closed form by expanding the series and resumming the terms, analogous to the calculation in [10]. However, since the singularity at $s = 0$ has to cancel, and the difference can only be meromorphic functions that fall off at infinity, it is simple to see that the real part is restricted to the trivial analytical continuation and the Laurent series. The functions $\tilde{w}$ generally have a large negative value, leading to a negative mass shift. This is mainly due to the large high-energy tail $\tilde{w}_{\frac{2n+1}{2}k} \sim 1/\sqrt{s}$ or $\tilde{w}_{nk} \sim 1/s$. This leading order behavior is necessary to be able to fit arbitrary $W(E)$ dependence, however, might be suppressed in the actual scattering, with an equal reduction of the mass shift.
Using these functions to solve the Lipmann-Schwinger equation in closed form effectively means making a simultaneous bubble sum approximation in each of the decay channels for each of the resonances. Therefore we refer to this method as the coherent bubble sum approximation. Single bubble sum approximations have be used widely. It corresponds to dressing the propagator of the resonance with its single or multiple decay channels. However, the case of an unitary, analytical, and covariant approach with channel mixing has only appeared at the heart of a few coupled-channel analyses with fixed form factors. [7–9] The difference is conceptually rather than technically. In the case of a single resonance, the bubble sum can be seen as a dressing of the resonance particle; a change in the particle properties. In the case of coupled channels such an interpretation does not exist. In this case only the physical observables in the scattering experiment are free of ambiguities in the interpretation. In simple terms, the resonance properties are intertwined and co-dependent.
Solving the Lipmann-Schwinger equation Eq. (2) reduces to solving the algebraic equation Eq. (3), and determining the Laurent-Taylor expansions, which is automated in the BUBBLEGUM code.

III. MASS SHIFTS

The K-matrix method leaves the real part of the resonance pole in the same position. The coherent bubble sum shifts the pole in order to satisfy analyticity contraints. These shifts can be substantial. However, the word "mass shifts" suggest that all effects can be reabsorbed in shifting the mass such that it accounts for main differences between K-matrix and T-matrix results. This is generally not the case. In a coupled channel problem, the mass shifts differ from channel to channel. In the K-matrix the real part of the pole, *i.e.*, the position where the real part of the amplitude crosses the axis, is at the same location in every channel as long as the coupling constant is not too large. For large coupling constants unitary yields stringent constraints, which can shift theses locations independently. For the exact result this does not hold. The isobar model, [3] which implies resonance properties independent of the decay channels, is therefore flawed from the start. The procedure of finite mass renormalization is not unique.

However, in order to make the comparison between K-matrix and exact result more fair for the K-matrix method, one can make a global mass shift ΔM_a for resonance a, which, in some way, averages the shifts in each of the channel:

$$\Delta M_a = \lim_{E \to M_a} \frac{(E - M_a)(1 - X_{11}\mathcal{W}_1 - X_{22}\mathcal{W}_2 + \det[\mathrm{X}]\mathcal{W}_1\mathcal{W}_2)}{\frac{\partial}{\partial M_a}(E - M_a)(1 - X_{11}\mathcal{W}_1 - X_{22}\mathcal{W}_2 + \det[\mathrm{X}]\mathcal{W}_1\mathcal{W}_2)} \quad , \tag{23}$$

where the expression is given by the common denominator in all of the T-matrix channels, Eq. (7-10). The term in denominator corresponds to the appropriate wave function renormalization. In the case of small coupled-channel effects and small overlap of the resonances the real part of the shifted pole positions coincide with the expected channel-independent pole position. However, in the case large couplings or coupled-channel effects this independent resonance mass shift breaks down, *i.e.*, linear approximation of the denominator around the pole is no longer valid. In that case an iterative scheme to recover the mass shifts is required. However, as the mass shift is not an invariant quantity, the results from such a scheme would have no real significance. Eventually one has to accept that the scattering data and the bare properties and couplings are the only well-defined quantities. However, the bare masses must always be given in combination with the coupling constants and the form factors.

IV. t-EXHANGE WITHOUT OPEN CHANNELS

This method of comparing scattering data to resonance properties can also serve as an approximate method for handling additional t-exchange processes. For low and medium energy scattering, t-exchange diagrams will not go on-shell, *i.e.*, there is no imaginary part associated with with the perturbative diagram $T^{(0)}_{\mathrm{exch}}$. However, the real part is a direct consequence of the energies at which the t-exchange diagram goes on-shell and strength in the channel. Therefore, the contributions can be approximated by a single resonance:

$$\begin{aligned}
\mathrm{Im}T^{(0)}_{\mathrm{exch}} &= -\int \mu(E')dE' g_{ix} f_i(E,E')\sqrt{\Gamma_i(E)}\pi\delta(E-E')\sqrt{\Gamma_j(E)} f_j(E,E') g^*_{jx} \\
&\to -\overline{\mu} g_{ix} f_i(E,M_{\mathrm{eff}})\sqrt{\Gamma_i(E)}\pi\delta(E-M_{\mathrm{eff}})\sqrt{\Gamma_j(E)} f_j(E,M_{\mathrm{eff}}) g^*_{jx} \quad , &(24)\\
T^{(0)}_{\mathrm{exch}} &= \overline{\mu} g_{ix} f_i(E,M_{\mathrm{eff}})\sqrt{\Gamma_i(E)}\frac{1}{E - M_{\mathrm{eff}} + i\epsilon}\sqrt{\Gamma_j(E)} f_j(E,M_{\mathrm{eff}}) g^*_{jx} \quad , &(25)
\end{aligned}$$

where $\mu(E')$ is a measure of the kinematical range over which the t-exchange diagram can go on shell, and M_{eff} is the mean of that measure:

$$\begin{aligned}
\int \mu(E')dE' &= \overline{\mu} \quad , &(26)\\
\int \mu(E')dE'E' &= M_{\mathrm{eff}} \quad . &(27)
\end{aligned}$$

In the case where the t-exchange diagram has an imaginary part in the kinematical range of interest, this approximation will not hold. More appropriate methods for this problem are under investigation and will be reported later.
In effective field theory highly virtual intermediate states are replaced by effective contact interactions. The effective resonance described here does more than that. The effective resonance that replaces the virtual t-exchange incorporates also the leading energy dependence of the t-exchange contribution.

V. S_{11} RESONANCES

The S_{11} resonances at 1.535 GeV and 1.65 GeV are a good test case for the coupled channel analysis of overlapping resonances as the threshold of the ηN channel is in the same region. However, there is only limited data available for the ηN channel. We fit the data between threshold and 1.77 GeV with minimal model of two bare masses and four coupling constants. Beyond the resonance region there is still a substantial

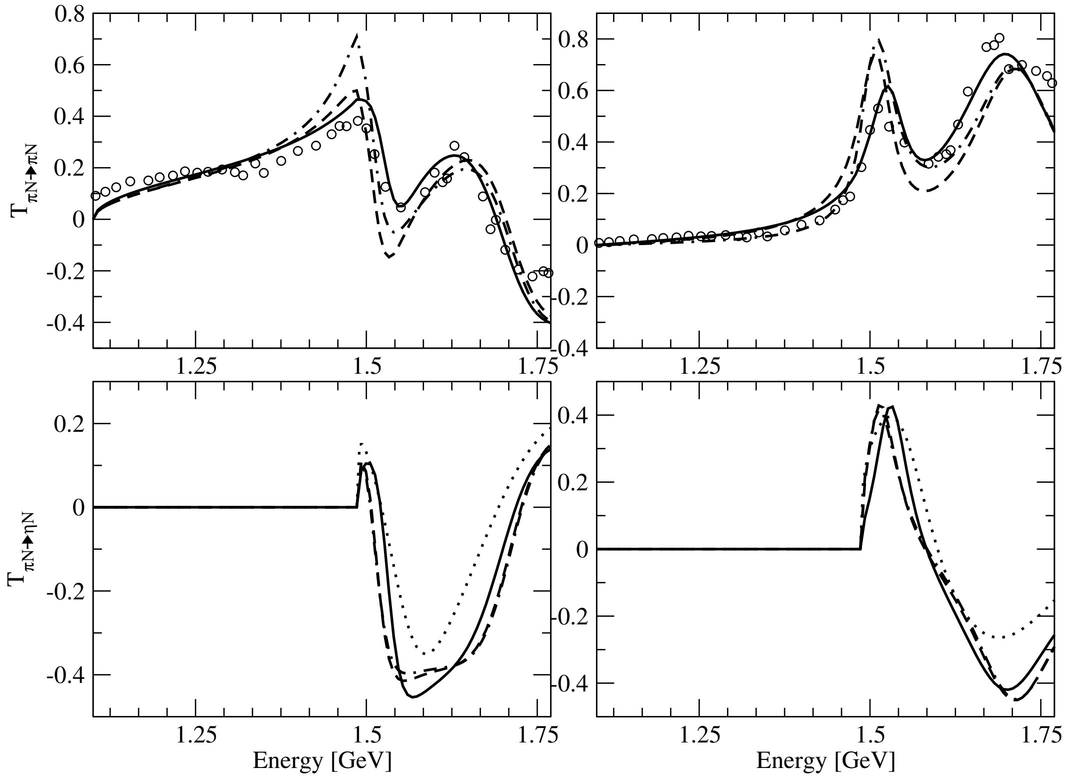

FIG. 1: The transition amplitudes in s-wave πN scattering. The top row are the real and imaginary part of the $\pi N \to \pi N$ amplitudes, and the bottom row the $\pi N \to \eta N$ amplitudes. The dots and circles are partial wave data, without error bars, the solid lines are the T-matrix results, the dashed lines the K-matrix approximation of these results, with identical parameters, and the dot-dashed lines the Breit-Wigner results, as described in the paper.

amplitude, and little data in other than the πN channel, which might indicate $\pi\pi N$ final states, and t-channel exchanges. A kink in the $\pi N \to \pi N$ data at 1.3 GeV is a further indication of $\pi\pi N$ states. For the clarity of the example we do not include the high-energy region and work with the minimal parameter set of six parameters. We use the GWU-VPI partial wave data[11] for the πN elastic amplitude and the results from the Pitt-ANL analysis[8] for the $\pi N \to \eta N$ channel. Standard fitting algorithms do not work properly as the variables are highly correlated in a non-trivial manner due to the unitarity and analyticity conditions. Instead we performed a global search, with an increasing mesh, zooming in on the optimal fit. We used version 1 of the BUBBLEGUM code, which does not perform global mass shifts, since these mass shifts can be erratic, causing problems for the fitting algorithm.

In Figure 1 we compare the exact T-matrix results with the K-matrix and Breit-Wigner approximations. The top left figure shows real part of the $\pi N \to \pi N$ transition amplitude, while the imaginary part appears on the right. The bottom row is the same combination for the $\pi N \to \eta N$ transition amplitude. The bare masses are 1.60 GeV and 1.79 GeV, while the renormalized masses come to 1.51 GeV and 1.71 GeV. The latter are used for the K-matrix and the Breit-Wigner approximations. The dots and the circles are the Pitt-ANL and GWU-VPI partial waves for the ηN and the πN channels respectively. The deviation for the ηN partial wave is due to the large error uncertainty, which led to the dominance of the πN channel in the fit. However, varying the relative importance of the ηN data, with respect to the πN data, did not significantly altered the results. It seems that the Pitt-ANL and GWU-VPI partial wave analysis, which include background terms, lead to an inconsistent ηN amplitude that cannot be fitted with with a resonance-only model like the coherent bubble sum approximation. With the current ηN data it is not possible to determine the nature of the deviations, and whether the data favors a background contribution. However, there seems to be no natural explanation for a background contribution. Somehow, the background should model some degrees of freedom, or states, that are important yet virtual, *i.e.*, not associated with open channels, such as, perhaps, the ρ-exchange between the πN pair.

Including the first term c_1 in the expansion Eq. (18) of the form factors halves the mean square deviation between the data and the fit. Notably, it increases the effective width of the πN form factor, while it decreases the effective width of the ηN form factors, corresponding, respectively, to a positive and a negative expansion coefficient c_1.
Note that it would be possible to obtain better fits in the K-matrix approximation and the Breit-Wigner approximation. However, we prefer to point out the differences which are solely the result of the approximations made, keeping the model identical. In this case it is clear that the first resonance $S_{11}(1535)$ is overestimated in both approximations. The coupling constants are $g_{1,\pi N} = -0.11$, $g_{2,\pi N} = 0.175$, $g_{1,\eta N} = -0.10$, and $g_{2,\eta N} = -0.11$.

VI. CONCLUSIONS

The K-matrix approximation does not require the solution of the dispersion integral. Therefore, it is often used to incorporate more complex interactions. In the Coherent Bubble Sum Approximation it correspond to setting $\tilde{W}_i$ to zero. The Breit-Wigner approximation is generally not a well-defined procedure. Eventually it boils down to replacing the complex interaction by a sum of Breit-Wigner forms with partial widths for each of the decay channels in each of the resonances:

$$T_{ij} = \sum_b \frac{g_{ib}\sqrt{\Gamma_j(E)\Gamma_i(E)}g^*_{jb}}{E - M_b - \sum |g_{ib}|^2\Gamma_i(E)} \quad , \tag{28}$$

where, in principle, each of the coupling constants and Γ_i's are unknown and should be recovered from the analysis. However, given the $W(E)$'s and the coupling constants, the closest general analogy is given by $\Gamma_i(E) = W_i(E)$.
A feature of scattering data where approximate methods such as Breit-Wigner and K-matrix approximations yield erroneous results is the interaction between thresholds and subthreshold resonances. A resonance with a strong coupling to a channel with a threshold at a higher energy might lead to anomalous threshold behavior. Once a resonance has an energy close enough to the threshold energy, it might get attracted into the channel, giving either a full circle in the Argand plot, or a sharp spike in the real part of the transition amplitude. Neither feature is properly reproduced in the Breit-Wigner or the K-matrix method.
In this paper the form of $W(E)$ is not derived. Only simple assumptions about its threshold behavior are made. The energy dependence can be modeled by a polynomial in the variable $(s - s_{\rm th})/s$. This freedom is an essential part of the Coherent Bubble Sum Approximation. If possible, the data should determine the function $W(E)$. Its value should be extracted from the scattering data without model dependence. It was only assumed that $W(E)$ falls off at infinity, which is considered realistic for hadronic systems. Analyzing resonance scattering data and modeling transition matrix elements of hadronic states are two separate problems and should be treated as such. In future studies we will examine t-exchange, multi-particle final states, and field-theoretical approaches which require renormalization.

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