Discreteness of Area and Volume in Quantum Gravity

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Abstract

We study the operator that corresponds to the measurement of volume, in non-perturbative quantum gravity, and we compute its spectrum. The operator is constructed in the loop representation, via a regularization procedure; it is finite, background independent, and diffeomorphism-invariant, and therefore well defined on the space of diffeomorphism invariant states (knot states). We find that the spectrum of the volume of any physical region is discrete. A family of eigenstates are in one to one correspondence with the spin networks, which were introduced by Penrose in a different context. We compute the corresponding component of the spectrum, and exhibit the eigenvalues explicitly. The other eigenstates are related to a generalization of the spin networks, and their eigenvalues can be computed by diagonalizing finite dimensional matrices. Furthermore, we show that the eigenstates of the volume diagonalize also the area operator. We argue that the spectra of volume and area determined here can be considered as predictions of the loop-representation formulation of quantum gravity on the outcomes of (hypothetical) Planck-scale sensitive measurements of the geometry of space.

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1 Introduction

In spite of recent progress, research in quantum gravity [1] has produced few precise physical predictions, against which the theory might be, at least ion principle, experimentally tested. In this paper, we show that, under certain assumptions, predictions for the spectra of certain geometric quantities can be derived from the quantum theory of gravity in the formulation based on the loop representation [2, 3, 4, 5].[1] More precisely, we show that the operators that correspond to the measurements of the volume and area of appropriately defined regions can be defined in this formulation of quantum gravity. We are able to diagonalize these operators, and we find that the spectra fall into certain discrete series, some of which we are able to explicitly determine. Finally, we argue that these spectra admit an interpretation as physical predictions of the loop-representation approach to quantum gravity.

There are two main sources of difficulties that must be overcome in order to derive physical results from non-perturbative formulations of quantum gravity. The first of these is the quantum-field-theoretical problem of defining operator products in the absence of the background geometry on which conventional regularization schemes rely. The second is the fact that in a diffeomorphism-invariant theory physical observables express correlations amongst degrees of freedom and are therefore non-trivial functions of the elementary fields. These two difficulties are addressed in this paper, and the methods by which this is done are the main technical underpinnings of our results. We address the first of these difficulties by utilizing a recently developed technique for constructing diffeomorphism-invariant regularizations of certain operator products [11]. This technique allows us to construct area [3] and volume operators and to compute their spectra. We address the second difficulty by considering physically meaningful observables which are obtained by using dynamical matter fields to construct spacetime reference frames, an old idea recently discussed in [12, 13, 14, 15, 16, 17]. This strategy, combined with the non-dynamical character of quantities we are considering, allows us to endow the spectra we compute with a physical interpretation [13, 14].

The volume $V(R)$ of a three-dimensional spatial region R is determined by the spacetime metric; in the context of quantum relativistic gravitation, therefore, the volume expresses an observable function of the quantum gravitational field. The main work of this paper is the construction of the opera-

[1] See also [6, 7, 8]. For detailed, but not up to date, reviews see [9, 10, 11].

tor that represents the quantum observable $\hat{V}(R)$ in the loop representation, and the study its associated spectral problem.

A key point, which underlies everything that is done here is that both volume and area are non-local functions of the local metric or frame fields. Local quantities tend to be divergent or ill-defined in non-trivial generally covariant quantum field theories [3, 4, 11], a fact likely to be related to the non-renormalizability of the perturbative series. The regularization technique we employ for constructing the volume operator [11] exploits the non-locality of the volume in order to circumvent this non-renormalizability of local quantities. An unphysical background metric is introduced for the sake of the regularization, but the operator turns out to be independent of this background, in the limit in which the cut-off is removed. Furthermore, as a consequence [3, 4, 11], the operator is finite and diffeomorphism invariant. This strategy differs significantly from that involved in the definition of operator products in conventional quantum field theories on fixed background fields, and we expect that it might be of general interest for the construction of non-trivial diffeomorphism invariant quantum field theories in the several different contexts in which they arise.[2]

Our main result is that we find that the spectrum of the volume is discrete and that eigenstates and eigenvalues can be expressed in terms of the spin network calculus, a technology developed many years ago by Roger Penrose with a quite different purpose [18]. We compute one component of the spectrum explicitly (eq.(1) below) and define a straightforward strategy for computing the rest of it.

Penrose introduced the spin networks as an attempt to construct a quantum mechanical description of the geometry of space. A spin network is a trivalent graph Γ (a graph in which each node joins three links), in which a positive integer p_l (a "color"), is associated to every link l of the graph (see Figure 1), with certain restrictions on the colors at each node. As shown in [21], one may associate a quantum state of the gravitational field to each spin network; these states are interesting because they can be used to define a basis of linearly independent loop states.[3] Here we show that the quantum states of the gravitational field defined by the trivalent spin networks are eigenstate of the volume operator. The eigenvalues are given as follows. Let p_i, q_i, r_i be the colors of the links adjacent to the i-th node of the spin

[2] For various recent perspectives on diffeomorphism invariant quantum field theories see [19, 20], and reference therein.

[3] Arbitrary loop states corresponding to different loops are generally not linearly independent, due to the spinor, or $SU(2)$-Mandelstam, identities [2].

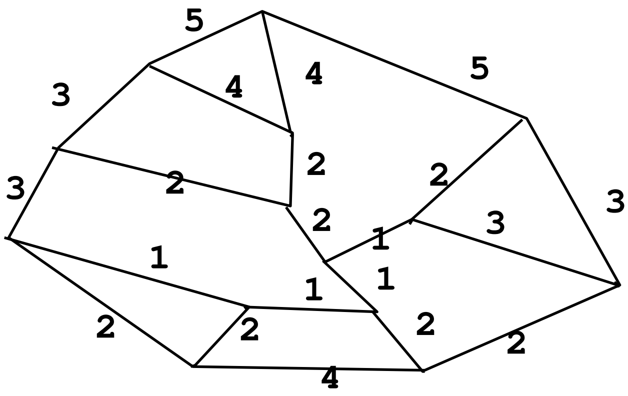

Figure 1: A spin network

network, and let a_i, b_i, c_i be defined by $p_i = a_i + b_i,\ q_i = b_i + c_i,\ r_i = c_i + a_i$. (The a_i, b_i, c_i are always integer due to the constraints on the coloring, and have a simple geometrical interpretation, which we will illustrate below.) We then find that the eigenvalues of the operator that corresponds to the volume of a region R are

$$V = \frac{1}{4}\, l_P^3 \ \sum_i \sqrt{a_i b_i c_i + a_i b_i + a_i c_i + b_i c_i}, \qquad (1)$$

where l_P is the Planck length and the sum runs over all the nodes i contained in R of an arbitrary spin network. This formula provides a component of the physical spectrum of the quantum volume.

Furthermore, we show here that the spin network states also diagonalize the operator $\hat{A}[\mathcal{S}]$, introduced in [3], which measures the area of the surface $\mathcal{S}$. (The two operators obviously commute.) We complete the diagonalization of the area, and we compute its full spectrum, which is

$$A = \frac{1}{2}\, l_P^2 \ \sum_l \sqrt{p_l^2 + 2p_l} \qquad (2)$$

where l labels the links of the spin network that cross the surface.

Given these results, we may ask whether there is there a relation between the spectra (1-2), and the volumes and areas that we physically observe. Physical information, including the information about geometry, resides in the diffeomorphism invariant observables[4]. If we assume that the universe is spatially compact, then the volume of a 3-dimensional spatial slice Σ of the universe is obviously invariant under 3-dimensional diffeomorphisms of Σ. If Σ is determined by the value of a physical field which is part of the dynamical theory considered (for instance if it is defined as a constant value surface of a scalar field), then this volume is also 4-dimensional diffeomorphism invariant, and hence physically meaningful.[5] In addition, if the theory includes also physical fields that determine finite spatial regions [12, 13, 14, 15, 16, 17], then the volumes of these regions too are 4-dimensional diffeomorphism invariant observables. The same holds for the area of surfaces defined by physical fields. Thus, in the presence of dynamical matter, area and volume of matter-determined surfaces are diffeomorphism invariant quantities. A very important observation is that *any* actual volume or area measurement that we may concretely perform defines such a 4-dimensional diffeomorphism invariant observable, because any region whose volume or area we measure is determined by physical fields and, in the context of relativistic gravitation, these fields are necessarily coupled to the gravitational field [12]. We shall argue below that the physical area and volume operators corresponding to those measurements in the context of a gravity-matter theory are the same mathematical operators as the pure gravity ones constructed here. It follows that the spectra are equal; and therefore the spectra we derive here can be interpreted as physical predictions from the quantum theory of gravity in the loop representation:

- *If one measured the volume of a physical region or the area of a physical surface with Planck scale accuracy, one would find that any measurement's result falls into the discrete spectra given here [as in eqs.(1-2)].*

In section 2 we construct the regularization procedure that yields the volume operator. This construction corrects certain difficulties of a previous definition [4, 11]. In section 3 we compute the action of the operator on spin network states, and we calculate its spectrum. To this purpose, we make

[4] For a discussion on the way in which purely gravitational diffeomorphism invariant observables (and loop-kind observables in particular) code the full physical information on spacetime geometry, see [22], where the 2+1 case, which is not plagued by the field-theoretical difficulties of the 3+1 case, is discussed.

[5] Under certain conditions the scalar field may play the role of a clock, and evolution with respect to such a clock field can be considered [4, 16, 17]; the volume operator can then be seen as a time dependent Heisenberg operator that represents the volume of the universe at a certain clock time.

use of the results and the technology developed in [21]. This technology simplifies the geometrical-combinatorial calculus on loop space, allowing the full computation of the action of the three-hands loop operator which enters the definition of the volume. In section 4 we study the area operator and derive its full spectrum. In section 5 we discuss the physical interpretation of the spectra. In section 6 we discuss the general result obtained.

2 Construction of the volume operator

Let us consider a three dimensional region R. The volume of this region is given by

$$V = \int_R d^3x \, \sqrt{\det g}, \tag{3}$$

where g is the three-dimensional metric of space. In terms of Ashtekar variables [23], the volume element can be written as

$$\det g = \left|\det \tilde{E}\right| = \frac{1}{3!} \left|\epsilon_{abc}\epsilon_{ijk}\tilde{E}^{ai}\tilde{E}^{bj}\tilde{E}^{ck}\right|. \tag{4}$$

Indices $a, b, ... = 1, 2, 3$ label tangent-space vector components and indices $i, j, ... = 1, 2, 3$ internal space components; we refer to [9, 10, 11] for notation and background.

If we try to construct the local quantum field operator corresponding to the volume element $\sqrt{\det g}$, we encounter the usual problem of making sense of a non linear function of operator-valued distributions. Here, $\sqrt{\det g}$ involves the square root of the cube of the field operator $\tilde{E}^{ai}(\vec{x})$. As in any quantum field theory, one may try to circumvent (undefined) products of operator-valued distributions by means of a regularization procedure. In the present context, however, the necessity of preserving diffeomorphism covariance frustrates the attempts to renormalize $\sqrt{\det g}$, as we shall discuss in a moment. The strategy that works is to sidestep the definition of the quantum operator $\sqrt{\det g}$ by constructing a regularization procedure that yields directly the non-local operator $\hat{V}$. This strategy was successfully introduced in [3] for the area operator, and is based on the (non-local) loop operators of the extended loop algebra [2].

Consider the three-indices ("three-hands") loop observable [2]

$$T^{abc}[\alpha](s,t,r) = Tr\left\{\tilde{E}^a(\alpha(s))\,U_\alpha(s,t)\,\tilde{E}^b(\alpha(t))\,U_\alpha(t,r)\,\tilde{E}^c(\alpha(r))\,U_\alpha(r,s)\right\}, \tag{5}$$

where $U_\alpha(s,t)$ is the parallel propagator of the Ashtekar connection along the loop α from the value s to the value t of the loop parameter, and $\tilde{E}^a = 4\tilde{E}^{ai}\tau^i$, where the generators of $SU(2)$ are given by $\tau^i = -\frac{\imath}{2}\sigma^i$, and σ^i are the Pauli matrices. Using the identity

$$Tr(\tau^i\tau^j\tau^k) = -\frac{1}{4}\epsilon^{ijk}, \tag{6}$$

we notice that in the limit in which the loop α shrinks to a point x we have

$$\lim_{\alpha\to x}\epsilon_{abc}T^{abc}[\alpha](s,t,r) = -16\ 3!\ \det\tilde{E}. \tag{7}$$

Trying to implement this limit in the quantum theory yields an uncontrolable divergence: in fact, if we renormalize the operator so that the limit remains finite, we find that the free renormalization "constant" produced by the renormalization transforms under diffeomorphisms as a scalar density [3]. This fact amounts to an infinite-fold renormalization ambiguity, because there is no preferred scalar density field on a differentiable manifold without background metric structure. The local quantum operator $\sqrt{\det g}$ is thus ill-defined in the theory. Notice that this difficulty is a direct consequence of the fact that the quantum field theory is to be defined on a differential manifold, namely on a much weaker structure than the metric manifolds on which quantum field theories are conventionally defined. We are indeed confronting here the very core of the problem of quantum gravity: making sense of a general covariant quantum field theory.

As we mentioned already, we deal with this difficulty by representing directly the volume of a *finite* region via a limiting procedure that does not require the (ill-defined) local operator $\sqrt{\det g}$ as an intermediate step. To this aim, let us fix an arbitrary (unphysical) flat metric g^0 in R, and let us partition the region R into cubic boxes (cubic with respect to g^0), whose sides have length L (with respect to g^0). We shall later verify that our results are independent from g^0 and from the partition chosen. We label the boxes with an index I. Clearly

$$V = \sum_I V_I, \tag{8}$$

where V_I is the volume of the I-th box. If L is small enough, by the very definition of the Riemann integral in (3), V_I is approximated by $L^3\sqrt{\det g(x_I)}$, where x_I is any point in the I-th box, and

$$V = \lim_{L\to 0}\sum_I L^3\sqrt{\left|\det\tilde{E}(x_I)\right|}. \tag{9}$$

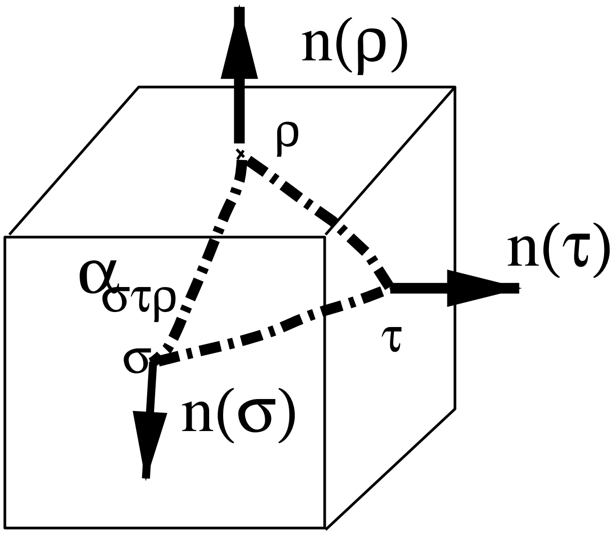

Figure 2: The $\hat{T}^{abc}$ operator

We now replace $\det \tilde{E}(x_I)$ by a non-local quantity that approximates it for small L. In the quantum theory this non-local quantity will remain finite for any finite L. Consider one particular box, and let ∂I indicate its boundary, namely the union of the six faces of the cube, oriented outwards. Consider three arbitrary points σ, τ and ρ on this boundary. Given these three points, let $\alpha_{\sigma\tau\rho}$ be the triangular loop formed by the three straight (in the metric g^0) segments that connect the three points σ, τ and ρ. We define

$$\mathcal{W}_I \equiv \int_{\partial I} d^2\sigma \int_{\partial I} d^2\tau \int_{\partial I} d^2\rho \ \left| n_a(\sigma) n_b(\tau) n_c(\rho)\, T^{abc}[\alpha_{\sigma\tau\rho}](s,t,r)\right|, \qquad (10)$$

where n_a is the cube's boundary outward-pointing normal vector (more precisely, it is the one-form tangent to the boundary of the box) and the integration measure is the area element induced by g^0. Furthermore, s, t and r are the values of the $\alpha_{\sigma\tau\rho}$ loop-parameter at the three vertices of the triangle, namely $\alpha_{\sigma\tau\rho}(s) = \sigma$, $\alpha_{\sigma\tau\rho}(t) = \tau$, $\alpha_{\sigma\tau\rho}(r) = \rho$: in other words, the hands of the loop observable sit on the box's boundary at the corners of the triangular loop. See Figure 2.

Then it is easy to verify that

$$\mathcal{W}_I = 2^7 3!\, L^6 \ \left|\det \tilde{E}\right| + O(L^7). \qquad (11)$$

The factor 2^7 comes from the 16 in equation (7) and from the fact that the triple integral on orthogonal faces of the cube yields a $(2L^2)^3$ term, since there are 2 faces of area L^2 for every orientation. The quantity $(2^7 3! L^6)^{-1}\mathcal{W}_I$ is thus a non-local quantity that approximates $\det \tilde{E}(x_I)$ for small L. Bringing everything together, we can write the total volume as the limit

$$V = \lim_{L\to 0} \sum_I \sqrt{\tfrac{1}{2^7 3!}\ \mathcal{W}_I}. \tag{12}$$

Notice that this expression is a $SU(2)$-gauge-invariant point splitting of the cubic product in the definition of the volume. Equations (10) and (12) may not provide the easiest way of calculating a volume classically, but, as we shall see, the sum in (12) can be promoted to an operator which is well-defined in the limit, in spite of the fact that the local volume element is ill-defined. Thus, we define the quantum operator corresponding to the physical volume of the region R as

$$\hat{V} = \lim_{L\to 0} \sum_I \sqrt{\tfrac{1}{2^7 3!}\ \hat{\mathcal{W}}_I}, \tag{13}$$

where $\hat{\mathcal{W}}_I$ is given by (10) with T^{abc} replaced by the corresponding operator $\hat{T}^{abc}$. What is going to happen is that the quantum $(\det \tilde{E}(x_I))$ diverges as L^{-6} (and thus in a badly background dependent way, because there is no covariant meaning in giving the same size to all boxes), therefore $\hat{\mathcal{W}}_I$ is finite in the limit; in addition, however, all but a finite number of terms in the sum (13) will vanish in the limit, so that (13) turns out to be finite.

The definition (13) of the quantum operator $\hat{V}$ requires us to specify the topology in which the limit is taken [11]. We take here the weak operator topology (namely convergence in the "matrix elements" $\langle\alpha|\hat{V}|\psi\rangle$), and we define the operator on the domain $\mathcal{D}$ formed by the loop functionals $\psi(\alpha) = \langle\alpha|\psi\rangle$ continuous in the loop space topology naturally induced by the manifold topology. (For an arbitrary smooth metric d on the manifold, the ϵ-neigborhood of α is formed by the loops β that admit a parametrization in which $d(\beta(s) - \alpha(s)) < \epsilon$ for all s.) Once defined on this domain, the operator can then be extended to its maximal domain $\mathcal{D}_{\max}$. This is a natural choice[6],

[6]As is well known, in promoting composite observables to quantum operators there might remain a degree of arbitrariness not entirely constrained by symmetry or consistency: quantization is an inverse problem, which may have more than one solution. The task here is to understand if there is *a* consistent quantum theory of gravity and to unravel its physical consequences, not to find *the* incontrovertibly correct one out of pure thought.

but we should point out that the issue is delicate, because the diffeomorphism invariant states are in $\mathcal{D}_{\max}$, but not in $\mathcal{D}$. The mathematical-physics technology which is being developed by Ashtekar, Isham and collaborators [6, 7] should allow to clarify these subtelties; note for instance that there are other relevant topologies in state space, as the norm topologies defined by the kinematical or diffeomorphism-invariant Hilbert structures [6, 7], which are inequivalent to the one in which the limit (13) is taken. See section 4.2 of [11] for a discussion of this point.

For completeness, let us recall from [2] the definition of the three-hand operator $\hat{T}^{abc}$:

$$\begin{aligned} \langle\beta|\hat{T}^{abc}[\alpha](s,t,r) &= l_P^6\ \Delta^a[\beta,\alpha(s)]\Delta^b[\beta,\alpha(t)]\Delta^c[\beta,\alpha(r)] \\ &\quad \times \sum_{j=1,8}(-1)^{r(j)}\langle(\alpha\#_{str}\beta)^j|. \end{aligned} \tag{14}$$

The distributional factor $\Delta^a[\beta,x]$ is

$$\Delta^a[\beta,x] = \int du\ \dot{\beta}^a(u)\ \delta^3(\beta(u),x) \tag{15}$$

and j labels the eight different loops $(\alpha\#_{str}\beta)^j$ obtained combining α and β by breaking and rejoining them in all possible ways at the three points of intersection s,t,r, where the operator acts (which we call "grasps"). The sign factor $r(j)$ is defined in [2] as the number of times that the orientation of loops' segments (coming from an arbitrary orientation of α and β) must be switched to yield a consistent routing; this cumbersome way of keeping track of the signs is superseded by the much simpler Penrose notation introduced in [21], which we shall recall and use below.

In the equation (14) the conventional bra notation of the loop representation is used (the loop functionals notation is obtained by contracting with a ket on the right: $\langle\alpha|\hat{O}|\psi\rangle = \hat{O}\psi(\alpha)$.) In this paper we shall find the left eigenvectors of $\hat{V}$. To the extent the theory is consistent (which is not proven), $\hat{V}$ must be hermitian and left and right spectra should coincide. Also, for notational simplicity, we shall from now on reverse the convention and write all the loop states as kets (so equation (14) becomes $\hat{T}^{abc}[\alpha](s,t,r)|\beta\rangle = \ldots$).

3 Action of the volume operator

Our task is to compute the action of the operator $\hat{V}$, given in eq.(13), on arbitrary quantum states. To this aim, we chose to work in a particular basis: the spin network basis [21]. As we will see, this basis essentially diagonalises the operators that we are considering; this will allows us to deal with the absolute value and the square route, which otherwise could have been sources of substantial difficulties.

We recall here the main elements of the definition of the spin network states [21]. A set of n fully overlapping loop segments is denoted as an n-rope, or a rope of degree n. An intersection between loops is called k-valent if k ropes (of any degree) emerge from it, and a loop state $|\gamma\rangle$ is called k-valent if γ contains intersections of valence at most k. We will begin by studying the action of $\hat{V}$ on trivalent states, and consider higher order intersections at the end of this section. A basis for the trivalent loop states is given by the spin network states. An (imbedded) spin network (Γ, p_l), is a trivalent graph Γ, imbedded in the spatial manifold Σ (which we take here compact and with fixed topology, say S^3), in which each link l is colored by a positive integer p_l, namely a positive integer is assigned to every link. At each node i of the graph, the colors of the three links adjacent to the node satisfy two relations: their total sum is even, and none is larger than the sum of the other two. A spin network quantum state $|\Gamma, p_l\rangle$ is obtained by a given spin network (Γ, p_l) as follows: First, replace every link l with a rope of degree p_l. Then, at each node join all the segments of the three adjacent ropes pairwise, in such a way that each segment is joined with a segment of one of the other two ropes. The conditions on the coloring make the matching always possible. Since all segments' end are joined, we obtain in this way a (multiple) loop $\gamma^{(\Gamma,p_l)}$. Consider the corresponding loop state $|\gamma^{(\Gamma,p_l)}\rangle$. Then, consider all possible permutations of the joining of all segments along every rope, so that the original choice of pairing becomes irrelevant. In this way, one obtains a family of $M = \prod_l p_l!$ (multiple) loops, which we denote as $\gamma_m^{(\Gamma,p_l)}$, $m = 1, ..., M$, or, if the context is clear, simply as γ_m. Each one of these loops have support Γ. The quantum state $|\Gamma, p_l\rangle$ associated to the spin network (Γ, p_l) is defined as

$$|\Gamma, p_l\rangle \;=\; \sum_m \epsilon_m \; |\gamma_m^{(\Gamma,p_l)}\rangle. \tag{16}$$

$$\epsilon_m \;=\; (-1)^{(p_m+n_m)} \tag{17}$$

where n_m is the number of single loops in the multiple loop $\gamma_m^{(\Gamma,p_l)}$ and p_m is the parity of the segments' permutation defining $\gamma_m^{(\Gamma,p_l)}$ (the overall sign, irrelevant for what follows, is not determined by this definition; see [21]).

One may think of a spin network state as the loop transform of the connection-representation state obtained by replacing every link colored p

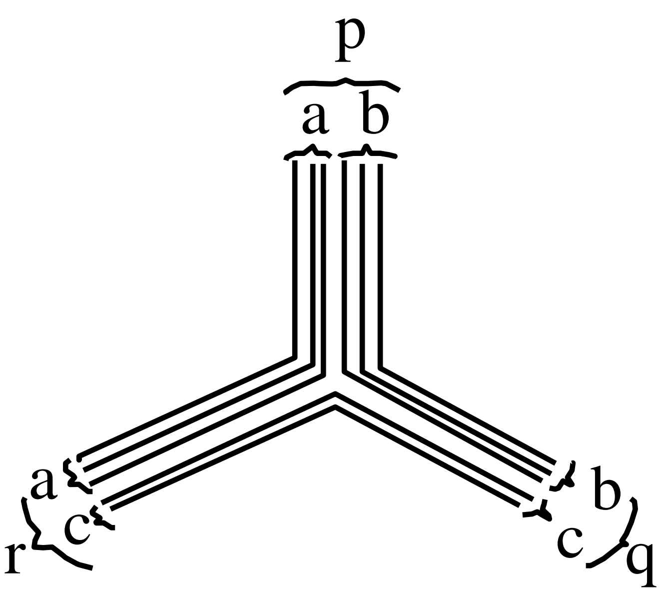

Figure 3: The integers a, b and c

by parallel propagators of the Ashtekar connection, taken in the spin $p/2$ representation, and then contracting the three parallel propagator matrices at each node in the unique $SU(2)$-invariant way, namely using the $6j$ symbols. The conditions on the colorings at each node then reflect the algebra of the $SU(2)$ representations, namely the angular momentum addition rules.

Consider the i-th node of a given spin network, and let p, q and r be the colors of the three adjacent links. We define a_i, b_i, c_i by

$$p = a_i + b_i, \quad q = b_i + c_i, \quad r = c_i + a_i. \tag{18}$$

When clear from the context, we will drop the suffix i and write just a, b, c. In constructing the loop state from the spin network, a_i is the number of individual loops that are routed through the i-th node from the link colored p to the link colored r, and so on, as shown by Figure 3. The numbers a_i, b_i, c_i provide a way of coding the coloring, but they are not independent: in a spin network with N nodes the $3N$ numbers a_i, b_i, c_i are related to each other because if a link (of color p) connects the i-th and j-th node then $p = a_i + b_i = a_j + b_j$, and so on for each link of the spin network; there is one linear relation of this kind for every link. We shall use also the notation $(\Gamma, a_i b_i c_i)$ for a spin network, and the notation $|\Gamma, a_i b_i c_i\rangle$ for the corresponding quantum state. For later convenience, we introduce also the following notation for the points along the ropes and their tangent: we parametrize each link l of the graph with a parameter s, and we denote the points of the links as $l(s)$ and their tangents as $\dot{l}^a(s)$. We will use $l(s)$ and $\dot{l}^a(s)$ in place of more cumbersome expressions such as $\gamma_m^{(\Gamma,p_l)}(s)$ and $(\dot{\gamma}_m^{(\Gamma,p_l)})^a(s)$ when the context is clear.

We want to compute the action of the volume operator on a (trivalent) spin network state. Using the definitions (13), and (10), we have

$$\begin{aligned}
\hat{V}\ |\Gamma, a_i, b_i, c_i\rangle &= \lim_{L\to 0}\sum_I \sqrt{\tfrac{1}{2^7 3!}\ \hat{\mathcal{W}}_I}\ \ |\Gamma, a_i b_i c_i\rangle \\
&= \lim_{L\to 0}\sum_I \Big(\tfrac{1}{2^7 3!}\int_{\partial I\times\partial I\times\partial I} d^2\sigma\, d^2\tau\, d^2\rho \\
&\quad \times \Big| n_a(\sigma) n_b(\tau) n_c(\rho)\ \hat{T}^{abc}[\alpha_{\sigma\tau\rho}](s,t,r)\Big|\ \Big)^{1/2} \\
&\quad \times |\Gamma, a_i b_i c_i\rangle. \qquad (19)
\end{aligned}$$

Let us fix a box I and three points σ, τ, ρ on its boundary, and let us consider the action of the operator valued distribution $\hat{T}^{abc}[\alpha_{\sigma\tau\rho}](s,t,r)$ on $|\Gamma, a_i b_i c_i\rangle$. Using the definition (16) of a spin network state and the definition (14) of the three-hands loop operator, we have

$$\begin{aligned}
&\hat{T}^{abc}[\alpha_{\sigma\tau\rho}](s,t,r)\,|\Gamma, a_i b_i c_i\rangle = \hat{T}^{abc}[\alpha_{\sigma\tau\rho}](s,t,r)\ \sum_m \epsilon_m |\gamma_m\rangle \\
&= \sum_m \epsilon_m \int ds'\ \dot{\gamma}_m^a(s')\,\delta^3(\gamma_m(s'),\sigma)\int dt'\ \dot{\gamma}_m^b(t')\,\delta^3(\gamma_m(t'),\tau) \\
&\times \int dr'\ \dot{\gamma}_m^c(r')\,\delta^3(\gamma_m(r'),\rho)\ \sum_{j=1,8} |(\alpha_{\sigma\tau\rho}\#_{(str)}\gamma_m)^j\rangle \qquad (20)
\end{aligned}$$

(here, clearly, γ_m means $\gamma_m^{(\Gamma,a_ib_ic_i)}$). The right hand side of this equation vanishes except in the case in which the three hands of the operator intersect Γ. Therefore, we have a non vanishing result only if the three points σ, τ, ρ are intersections points between the boundary of the box and Γ. Let us assume this is the case. We denote the three ropes that intersect σ, τ, ρ by l_σ, l_τ, l_ρ respectively (they may coincide), and their degree by p_σ, p_τ and p_ρ. The three ropes are thus formed by p_σ, p_τ and p_ρ segments respectively, each one of which is seen by one of the hands of the operator. Let us introduce an index S, running from 1 to p_σ that labels the p_σ segments in the rope

l_σ, and, similarly, indices $T = 1...p_\tau$ and $R = 1...p_\rho$ labeling the segments of l_τ and l_ρ respectively; and let us denote by s_S, t_T, r_R the values of the loop parameter corresponding to the intersections between the box boundary and the S-th, T-th and R-th segments of the rope l_σ, l_τ and l_ρ respectively. We have then

$$\begin{aligned}&\hat{T}^{abc}[\alpha_{\sigma\tau\rho}](s,t,r)\,|\Gamma, a_ib_ic_i\rangle = \\ &= \int ds\, \dot{l}^a_\sigma(s)\, \delta^3(l_\sigma(s),\sigma) \int dt\, \dot{l}^a_\tau(t)\, \delta^3(l_\tau(t),\tau) \int dr\, \dot{l}^a_\rho(u)\, \delta^3(l_\rho(r),\rho) \\ &\times|\Psi\rangle, \end{aligned} \tag{21}$$

where

$$|\Psi\rangle = \sum_m \; \epsilon_m \; \sum_S \sum_T \sum_R \sum_{j=1,8} |(\alpha_{\sigma\tau\rho}\#_{(s_S t_T r_R)}\gamma_m)^j\rangle. \tag{22}$$

We repeat for clarity: m labels the multiple loop states that form the spin network state, or, equivalently, labels all the possible symmetrizations of all the segments along each rope of the net; S, T and R label the segments in the three ropes that intersect the box; once fixed one of these segments in each of the three ropes, j labels the eight outcomes of the three graspings of the operator on these three segments.

Now, the key observation is the following. First, let us exchange the order of the summations in the last equation by summing over m first. Then, consider what happens to Ψ in the limit in which the box size L goes to zero. The loop $(\alpha_{\sigma\tau\rho}\#_{(s_S t_T r_R)}\gamma_m)^j$ is obtained by joining the spin network loop $\gamma_m^{(\Gamma,a_ib_ic_i)}$ with the triangular loop $\alpha_{\sigma\tau\rho}$ which has size L. In the limit in which L is zero the three grasping points σ, τ, ρ overlap and $\alpha_{\sigma\tau\rho}$ is a single-point loop, located, say, in the point P_I. In this limit, the loop $(\alpha_{\sigma\tau\rho}\#_{(s_S t_T r_R)}\gamma_m^{(\Gamma,a_ib_ic_i)})^j$ has the same support as the grasped loop $\gamma_m^{(\Gamma,a_ib_ic_i)}$, namely Γ, and can differ from $\gamma_n^{(\Gamma,a_ib_ic_i)}$ only by a rearrangement of the joining of the segments in the point P_I. But by rearranging the way the segments are joined, we can only obtain one of the other loop states of the spin network, say $|\gamma_{m'}^{\Gamma,a_ib_ic_i}\rangle$ for a certain m', because the spin network state is the linear combination of *all* loop states obtained symmetrizing the ropes in all possible ways. Using the continuity of the loops functionals in the domain of definition of $\hat{V}$ (see the end of section 3), we have thus necessarily

$$\sum_m \; \epsilon_m \; |(\alpha_{\sigma\tau\rho}\#_{(s_P t_Q u_R)}\gamma_m^{(\Gamma,a_ib_ic_i)})^j\rangle = c(S,T,R,j)|\Gamma, a_ib_ic_i\rangle + O(L). \tag{23}$$

In the next subsection we will verify this result explicitly and compute the proportionality coefficient $c(S,T,R,m)$. This reconstruction of the full spin network state in the limit is the technical reason for which the spin network states diagonalise the volume operator. Using this result, we have

$$\begin{aligned}|\Psi\rangle &= \sum_{STR}\sum_{j=1,8} c(S,T,R,j)|\Gamma, a_ib_ic_i\rangle + O(L) \\ &\equiv C(\Gamma, a_ib_ic_i; l_\sigma l_\tau l_\rho)\; |\Gamma, a_ib_ic_i\rangle + O(L),\end{aligned} \tag{24}$$

where we have explicitly indicated that the proportionality coefficient depends on the spin network and on the ropes grasped. Obtaining in the limit a state proportional to the one on which the operator acts is the result that allows us to navigate through the computation of the volume operator.

To be more precise, the point P_I can be either on a link or on a node. (If it is on neither, C is zero.) If it is on a link, it is easy to see that the rearrangement of the joining of the segments has no effect on a spin network, because the segments are symmetrized along the link anyway. But the same is true also if P_I is a node, because there is only one existing combination of fully symmetrized ropes in a trivalent nodes. This is a central property of the spin networks, which corresponds to the fact that there is a unique invariant way of combining three irreducible representations of $SU(2)$.

We will compute the coefficient of proportionality $C(\Gamma, a_i, b_i, c_i; l_\sigma, l_\tau, l_\rho)$ in the next subsection. For the moment, we use (21) and (24) to write (19) as

$$\begin{aligned}&\hat{V}\; |\Gamma, a_ib_ic_i\rangle = \lim_{L\to 0}\sum_I \Big(\tfrac{1}{2^7 3!}\int_{\partial I\times\partial I\times\partial I} d^2\sigma\, d^2\tau\, d^2\rho \\ &\times |n_a(\sigma)n_b(\tau)n_c(\rho) \sum_{l_\sigma l_\tau l_\rho} \\ &\times \int ds\, l^a_\sigma(s)\,\delta^3(l_\sigma(s),\sigma)\int dt\, l^a_\tau(t)\,\delta^3(l_\tau(t),\tau)\int dr\, l^a_\rho(r)\,\delta^3(l_\rho(r),\rho) \\ &\times C(\Gamma, a_i, b_i, c_i; l_\sigma l_\tau l_\rho)|\Big)^{-1/2} \quad |\Gamma, a_ib_ic_i\rangle.\end{aligned} \tag{25}$$

We have taken a delicate step here, by taking the limit of a term inside the integration. [7]

[7] In a rigorous treatement, one should check the validity of this step by keeping track of subleading terms in L and verifying that they remain of lower order also after the integration. This is not easy, since the operator is not diagonal beyond $L = 0$, which makes the computation of its square root more difficult.

Due to the delta functions, we can take the one-dimensional integrals outside the absolute value. Then, we use the well known fact that for every two dimensional surface Σ and every loop β

$$\int_{\Sigma} d^2\sigma \int ds \quad n_a(\sigma)\dot{\beta}^a(s)\ \delta^3(\sigma,\beta(s)) = I[\Sigma,\beta] \tag{26}$$

is just the (oriented) intersection number between the surface and the loop. This simple relation plays a key role in loop-representation regularization techniques, because it allows the action of the operators to be expressed in topological terms (intersection numbers), which then permits diffeomorphism invariance to be restored.[11] Since the absolute value of a product of three numbers each of which is either 1 or -1 can only be the unit, we have immediately that

$$\hat{V}|\Gamma,a_i,b_i,c_i\rangle = \lim_{L\to 0}\sum_I \sum_{l_\sigma l_\tau l_\rho} \sqrt{2^{-7}\,|C(\Gamma,a_i,b_i,c_i;l_\sigma l_\tau l_\rho)|}\ |\Gamma,a_ib_ic_i\rangle \tag{27}$$

where the sum $\sum_{l_\sigma l_\tau l_\rho}$ runs over all triples of ropes intersecting the boundary of the box I. Note that the factor $1/3!$ has been absorbed by the fact that there are 3! terms coming from the integrals for each given triple of intersections.

There is an important qualification to be added to the meaning of the sum $\sum_{l_1 l_2 l_3}$: the sum runs only over triples of *distinct* intersections; namely the three intersections between ropes and boundary must lay in three distinct points of the boundary of the box in order to give a non vanishing contribution. The reason is the following. In the derivation of (27) from (25), every triple of intersections gives rise to 3! terms, obtained by switching the locations of σ, τ and ρ. These terms are equal in magnitude, but have alternating signs, because $\hat{T}^{abc}$ changes sign by reversing the order of the hands. If the three intersection points are disjoint, then each one of the 3! terms comes from a different integration point of the triple (six-dimensional) integration space. Since we are taking the absolute value first and then integrating, the different signs of the terms are irrelevant: we have to sum their absolute values. However, if (at least) two of the intersection points overlap, then for each of the 3! terms there is a brother term (obtained by switching the two overlapping intersections), which is equal in magnitude, but opposite in sign. The two terms with opposite sign arise *at the same integration point.* Thus, we must take the absolute value *after* summing the two terms. Therefore they cancel each other.

The important consequence of the fact that only distinct intersections enter the sum (27) is that the action of the operator is zero unless the loop intersects the boundary of the box in at least three distinct points; a moment of reflection shows that for small enough L, this can happens only if there is a node inside the box. As $L \to 0$, we reach a point at which each box contains at most one node. Therefore we have the result that the sum over the boxes I (which are infinite in the limit) reduces to a sum over the nodes i contained in the region R (which are finite in the limit). So we can finally take the limit, and get

$$\hat{V}\ |\Gamma, a_i b_i c_i\rangle = \sum_i \sum_{l_\sigma l_\tau l_\rho} \sqrt{2^{-7}\,|C(\Gamma, a_i b_i c_i; l_\sigma l_\tau l_\rho)|}\ \ |\Gamma, a_i b_i c_i\rangle, \tag{28}$$

where now $l_\sigma l_\tau l_\rho$ are the triples of ropes adjacent to the i-th node. For a trivalent state there is only one triple of ropes emerging from every node, and therefore we can write

$$\hat{V}\ \ |\Gamma, a_i b_i c_i\rangle \ \ = \sum_i \sqrt{2^{-7}\,|C(\Gamma, a_i, b_i, c_i; i)|}\ \ |\Gamma, a_i b_i c_i\rangle.; \tag{29}$$

while we will use the expression (28) for non-trivalent states.

3.1 The combinatorics of the routings

In order to compute $C(\Gamma, a_i b_i c_i; l_\sigma l_\tau l_\rho)$, defined in (23) and (24), we use the Penrose notation. The present exercise is a demonstration of the power of this formalism. We must compute C from

$$\begin{aligned}\sum_{S=1,p_\sigma} \sum_{T=1,p_\tau} \sum_{R=1,p_\tau} \sum_{j=1,8} \sum_m \epsilon_m\ |(\alpha_{\sigma\tau\rho} \#_{(s_S t_T r_R)} \gamma_m^{(\Gamma, a_i b_i c_i)})^j\rangle &= \\ = C(\Gamma, a_i b_i c_i; l_\sigma l_\tau l_\rho)\ |\Gamma, a_i b_i c_i\rangle. \end{aligned} \tag{30}$$

Let the node we are dealing with be the i-th one. The sum includes $8 \times p_\sigma \times p_\tau \times p_\rho$ terms: we have one of the three hands of the operator per rope; each hand may grasp any of the segments in the rope, giving $p_\sigma \times p_\tau \times p_\rho$ triple graspings; and each triple grasping produces $2^3 = 8$ terms from the two possible outcomes of each grasp. The triple graspings produce different outcomes, according to how the grasped segments are rooted among themselves through the node. A short reflection shows that there are three basic possibilities that we list in Figure 4 (up to the obvious $2\pi/3$ rotational symmetry).

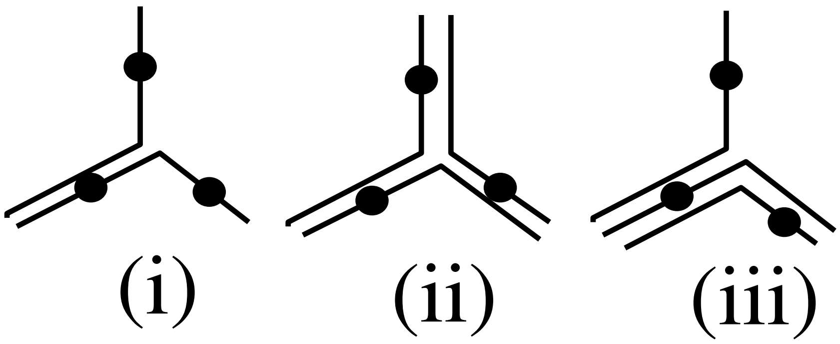

Figure 4: The three possible grasps

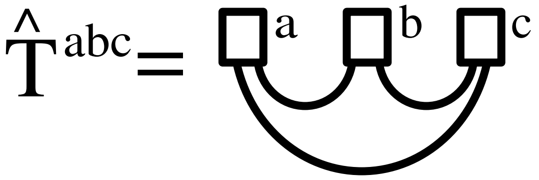

Figure 5: The diagrammatic representation of T^{abc}.

We denote these three possibilities as triple grasps of the first, second, and third kind. Counting all possible instances, the number of (triple) grasps of the first kind (i. in the Figure) that contribute to the sum (30) is $2abc$; the number of grasps of the second kind (ii) is $2(ab+bc+ca)$; and the number of grasps of the third kind (iii) is $[a(a-1)(b+c)+b(b-1)(a+c)+c(c-1)(b+a)]$. Here a, b and c are defined in terms of p_σ, p_τ and p_ρ as in (18), and we drop the suffix i for simplicity; we will reintroduce it when needed. One may verify that the total number of grasps of any kind is indeed $p_\sigma \times p_\tau \times p_\rho$.

Let us recall the basic rules of the diagrammatic Penrose calculus [21]. The $\hat{T}^{abc}$ operator is represented by the figure with three boxes shown in Figure 5; each box standing for a hand of the operator.

The elementary action of a box is given in Figure 6.

Let us begin by considering the grasp of the first kind (i). In diagrammatic notation, this is given in Figure 7, where the circles labeled $1, 2$ and 3 represent the three ropes emerging from the node.

The result of the grasp is computed using the basic rule Figure 6, and is given in in Figure 8.

Expanding the symmetrizations, we get the 8 terms of Figure 9. We see

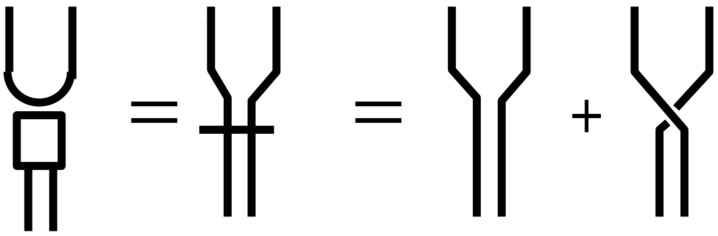

Figure 6: The grasp

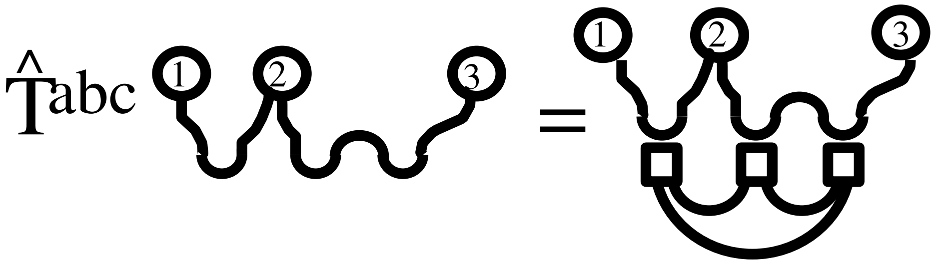

Figure 7: The grasp of the T^{abc} on the node of type (i)

immediately that the first, second and third of these terms vanish, because all the loops in the rope 2 are symmetrized and an identity of the spin calculus is Figure 10.

The remaining terms become, using the sign rules of [21] those shown in Figure 11, all proportional to the original state.

This confirms the expected result that the action of the volume is diagonal and provide the correct term in the eigenvalue. Recalling that there are $2(ab+bc+ca)$ terms of this kind in the grasping, we thus have a contribution $8(ab+bc+ca)$ to C from the graspings of the second kind. A completely analogous calculation, which we leave to the reader as a simple exercise of application of the rules of [21], shows that the contribution of the grasp of the first kind is $8abc$, while the contribution of the grasps of the third kind vanishes. Putting all these results together, we have

$$\begin{aligned}\sum_{STR}\sum_{j}\sum_{m}\epsilon_m|(\alpha_{\sigma\tau\rho}\#_{(s_S t_T r_R)}\gamma_m^{(\Gamma,a_ib_ic_i)})^j\rangle \\ = 2^3 l_P^6(abc+ab+bc+ca)\ |\Gamma,a_ib_ic_i\rangle. \qquad (31)\end{aligned}$$

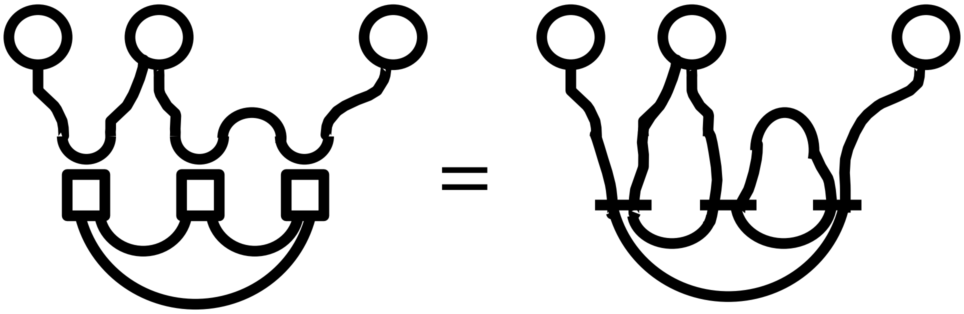

Figure 8: Action of the grasp of the T^{abc} on the node of type (ii)

Inserting in (29), we have finally

$$\hat{V} \ \ |\Gamma, a_i b_i c_i\rangle = \frac{1}{4} \ l_P^3 \ \sum_i \sqrt{|a_i b_i c + a_i b_i + b_i c_i + c_i a_i|} \ \ |\Gamma, a_i b_i c_i\rangle, \qquad (32)$$

where we restored the suffix i that indicates the node to which a, b and c refer. We recall that the sum is over all nodes i in the region R. In other words, the component of the spectrum of the volume of the region R corresponding to the trivalent states is the one given in (1).

Finally, let us consider the non-trivalent states. The nodes of order larger than 3 are not unique, thus in general the coloring p_l of the link are not sufficient to characterize the state and we need an additional set of integers v_i, one for every node i, to characterize the routing of the individual segments through the i-th node. We may thus indicate the states as $|\Gamma, p_l, v_i\rangle$. The range of the index v_i is finite, and depends on the order of the i-th node and on the coloring of the adjacent links. For instance, there are two independent 4-valent nodes with all 4 links colored 1; thus for such an node $v_i = 1, 2$.

If the point P_I, to which the I-th box is shrinking in the limit, coincides with the i-th node, and this node is not trivalent, then it is not true anymore that in the limit the grasping gives the original state back, precisely because nodes of order larger than 3 are not unique. A rearranging of the routing of the segments at the node may modify the state. Thus, equation (29) does not hold anymore. However, since the possible routings v_i through each node are of finite number, by repeating the same reasoning as in the trivalent case, we have in general that

$$\hat{V} \ |\Gamma, p_l, v_i\rangle = \sum_i \sqrt{2^{-7}|\hat{C}_i|} \ |\Gamma, p_l, v_i\rangle \qquad (33)$$

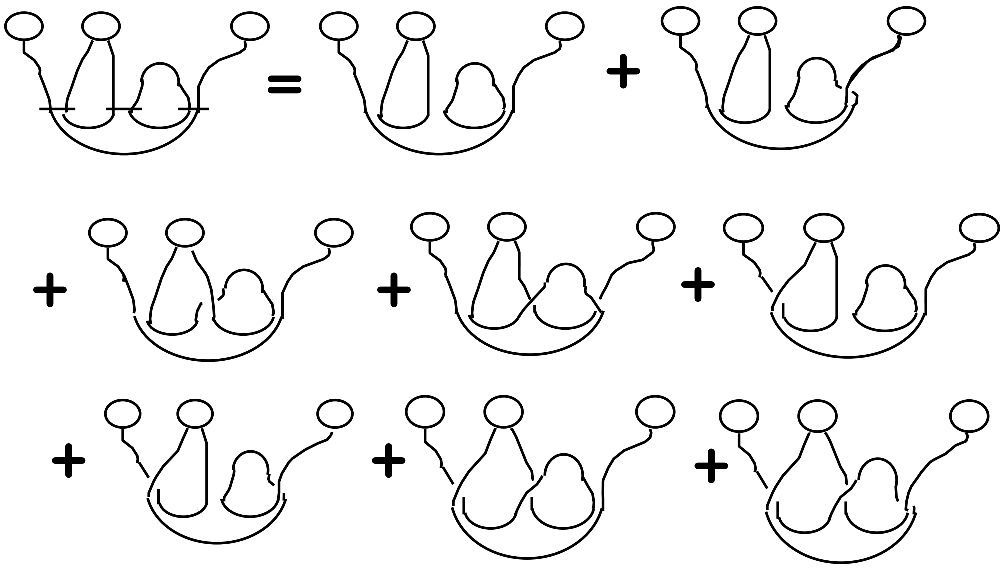

Figure 9: Expansion of the symmetrization

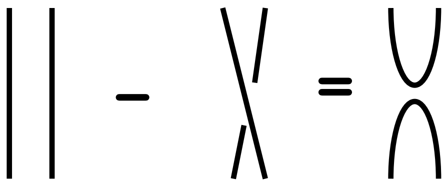

Figure 10: The basic spinor identity

where the operator $\hat{C}_i$

$$\hat{C}_i \; |\Gamma, p_l, v_i\rangle = \sum_{v_i'} C_{v_i}^{v_i'} \; |\Gamma, p_l, v_i'\rangle, \tag{34}$$

is a finite dimensional matrix, uniquely determined by the order of the i-th node and the coloring of its adjacent links. These matrices are defined by

$$\sum_{v_i'} C_{v_i}^{v_i'} \; |\Gamma, p_l, v_i'\rangle = \sum_m \epsilon_m \sum_{l_\sigma l_\tau l_\rho} \sum_{STR} \sum_{j=1,8} |(\alpha_{\sigma\tau\rho} \#_{(s_S t_T r_R)} \gamma_m^{(\Gamma,p_l)})^j\rangle \tag{35}$$

and can be computed using the Penrose calculus, as we did above.

Clearly, the eigenstates of the volume operator can be found by diagonalising the $C_{v_i}^{v_i'}$ matrices. If we denote by $c_{v_i}^{(n)}$ the eigenvectors and by $\lambda_i^{(n)}$

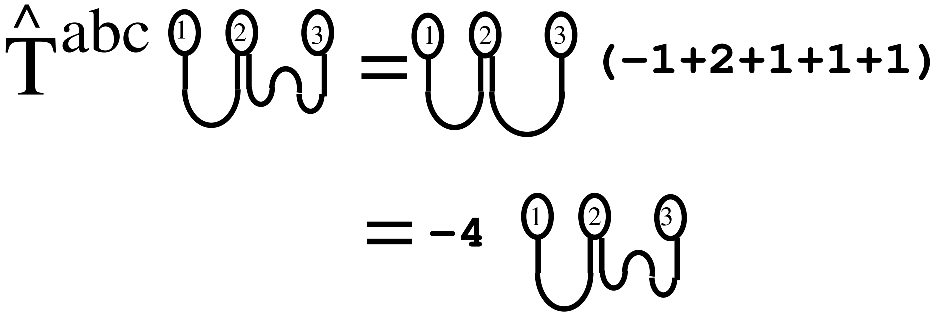

Figure 11: The result of the grasp on the node

the eigenvalues, so that $C^{v'_i}_{v_i}\ c^{(n)}_{v'_i} = \lambda^{(n)}_i\ c^{(n)}_{v_i}$, then it is immediate to verify that the states

$$|\Gamma, p_l, n_i\rangle = \sum_{v_1 \ldots v_I} c^{(n_1)}_{v_1} \ldots c^{(n_I)}_{v_I}\ |\Gamma, p_l, v_i\rangle \tag{36}$$

where I is the number of nodes in Γ, are eigenstates of the volume with eigenvalue

$$V = \sum_i \sqrt{2^{-7} |\lambda^{(n_i)}_i|}. \tag{37}$$

This formula gives the complete spectrum of the volume operator. To compute it explicitly, we must calculate and diagonalise the family of matrices $C^{v'_i}_{v_i}$.

We conclude this section with a most important observation. Consider the case in which the region R is given by the entire three dimensional space. In this case, it is immediate to verify that the quantum volume operator, defined in (33), satisfies

$$U(\phi)\hat{V} = \hat{V}U(\phi) \tag{38}$$

for any diffeomorphism ϕ of the three-manifold (in the connected component of the identity), where $U(\phi)$ is the generator of finite diffeomorphisms which acts on the quantum state by displacing loops [2]. This is an immediate consequence of the fact that the action of $\hat{V}$ is only sensitive to features of the loop state that are invariant under diffeomorphisms. It follows that $\hat{V}$ is well defined on the equivalence classes of quantum states under diffeomorphisms, namely on the knot states [21]. Thus $\hat{V}$ defines a genuine operator acting on knots states. This operator acts on the intersections of the knots, with no reference to position (or momentum) space. It represents an example of the combinatorial operators which may describe quantum general covariant physics on knot space.

4 Area

The operator $\hat{A}[\mathcal{S}]$ that corresponds to the area of a surface $\mathcal{S}$ was defined in [3], where a component of its spectrum was computed. This component corresponds to the eigenstates formed by states in which the surface $\mathcal{S}$ is pierced by single lines, or 1-ropes. The rest of the spectrum, which we are about to compute, corresponds to states in which the surface is crossed by ropes of degree higher than one.

The area of a surface $\mathcal{S}$ is given, in terms of the Ashtekar conjugate variable by [24]

$$A = \int_{\mathcal{S}} d^2\sigma \, \sqrt{\tilde{E}^{ai}\tilde{E}^{bi} \, n_a n_b} \tag{39}$$

The two-hands loop observable

$$T^{ab}[\alpha](s,t) = Tr[U_\alpha(s,t)\, \tilde{E}^a(\alpha(t))\, U_\alpha(t,s)\, \tilde{E}^b(\alpha(s))] \tag{40}$$

converges to $16\tilde{E}^{ai}(x)\tilde{E}^{bi}(x)$ when α shrinks to x. Therefore for a small surface $\mathcal{S}_I$ we can write

$$A_I^2 = \int_{\mathcal{S}_\mathcal{I}} d^2\sigma \int_{\mathcal{S}_\mathcal{I}} d^2\tau \left| \frac{1}{8} \, n_a(\sigma) n_b(\tau) T^{ab}[\alpha_{\sigma\tau}](s,t) \right| \tag{41}$$

up to small terms; here $\alpha_{\sigma\tau}$ is a small loop going through σ and τ. Following the same idea used for the volume, we partition the surface $\mathcal{S}$ into small square surfaces $\mathcal{S}_I$ of side L, so that we have

$$A = \lim_{L\to 0} \sum_I \sqrt{A_I^2}, \tag{42}$$

and we define the quantum area operator as

$$\hat{A} = \lim_{L\to 0} \sum_I \sqrt{\hat{A}_I^2}, \tag{43}$$

where $\hat{A}_I^2$ is defined by replacing T^{ab} with $\hat{T}^{ab}$ in (41).

Let us now compute the action of $\hat{A}$ on a spin network state. ¿From the definition of the two-hands loop operator [2] we have

$$\begin{aligned}\hat{A}_I^2\ |\Gamma, p_l\rangle = & \int_{\mathcal{S}_\mathcal{I}\times\mathcal{S}_\mathcal{I}} d^2\sigma\, d^2\tau \left| \frac{1}{8}\ n_a(\sigma) n_b(\tau) \sum_m \epsilon_m \int ds \int dt \right. \\ & \times \left. \dot{\gamma}_m^a(s)\, \delta^3(\gamma_m(s),\sigma)\ \dot{\gamma}_m^b(t)\, \delta^3(\gamma_m(t),\tau)\ n_a(\sigma) n_b(\tau) \right| \\ & \times \sum_{j=1,4} |(\alpha_{\sigma\tau} \#_{st} \gamma_m)^j\rangle. \qquad (44)\end{aligned}$$

The right hand side gets contributions only from the intersections between γ_m and the surface. For small enough L there will be at most one rope l crossing $\mathcal{S}_I$. Assume that this is the case and that the rope l has degree p, and label its segments by an index P. Following the same argument we used for the volume operator, we have

$$\begin{aligned}\hat{A}_I^2\ |\Gamma, p_l\rangle = & \int_{\mathcal{S}_\mathcal{I}\times\mathcal{S}_\mathcal{I}} d^2\sigma\, d^2\tau \\ & \times \left| \frac{1}{8}\ n_a(\sigma) n_b(\tau) \sum_m \epsilon_m \int ds \int dt \right. \\ & \times \left. \dot{l}^a(s)\, \delta^3(l(s),\sigma)\ \dot{l}^b(t)\, \delta^3(l(t),\tau) \right| \quad |\Psi\rangle \qquad (45)\end{aligned}$$

where

$$|\Psi\rangle = \sum_P \sum_{P'} \sum_{j=1,4} |(\alpha_{\sigma\tau} \#_{s_P t_{P'}} \gamma_m)^j\rangle = c(p)\ |\Gamma, p_l\rangle. \qquad (46)$$

(If the intersection point between the surface and the spin network is a node, then only the loops rooted through the node accross the surface contribute to the sum.) Inserting in the definition of the area operator we have

$$\begin{aligned}\hat{A}\ |\Gamma, p_l\rangle \ = \ & \lim_{L\to 0} \sum_I \left(\int_{\mathcal{S}_\mathcal{I}\times\mathcal{S}_\mathcal{I}} d^2\sigma\, d^2\tau \left| \frac{1}{8}\ n_a(\sigma) n_b(\tau) \int ds \int dt \right.\right. \\ & \times \left.\left. \dot{l}^a(s)\, \delta^3(l(s),\sigma)\ \dot{l}^b(t)\, \delta^3(l(t),\tau)\ c(p) \right| \right)^{-1/2} |\Psi\rangle. \quad (47)\end{aligned}$$

The integrals are immediate, giving

$$\hat{A}\ |\Gamma, p_l\rangle = \lim_{L\to 0} \sum_I \sqrt{\left| \frac{1}{8}\ c(p) \right|}\ \ |\Psi\rangle. \qquad (48)$$

Two kinds of terms come into the evaluation of this action. The combinatorial calculus is summarized in Figures 12 and 13. The result is

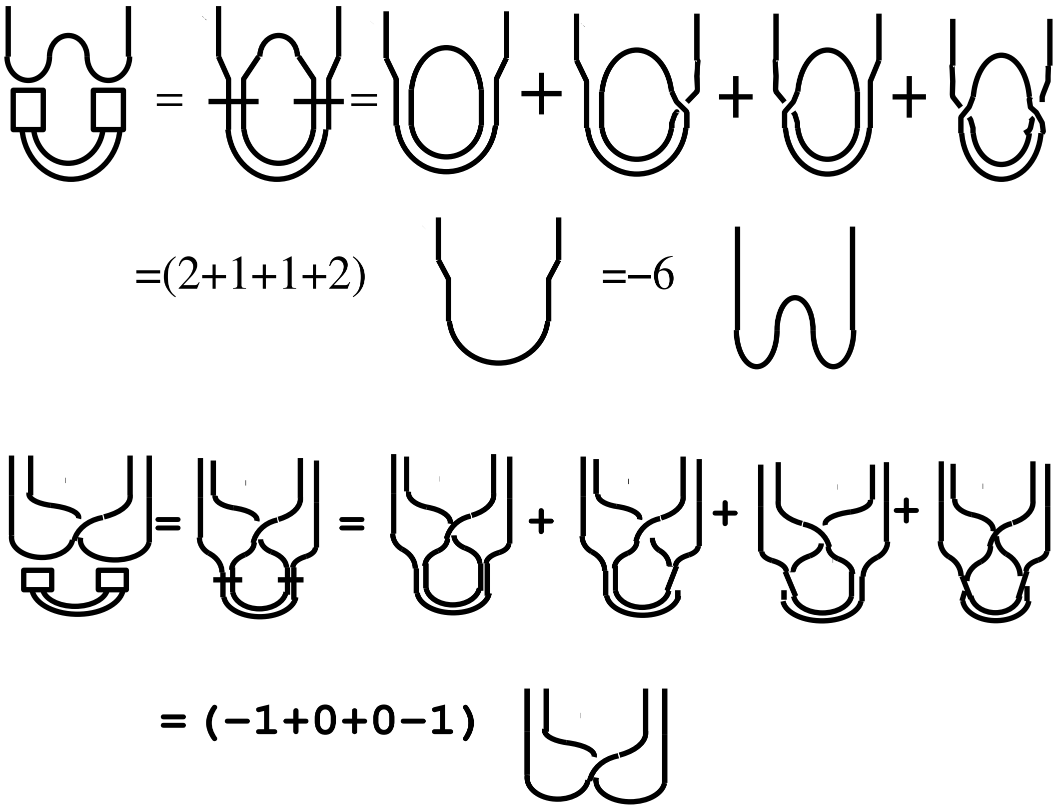

Figure 12: The two kinds of terms that come into the action of the area operator

$c(p) = -2(p^2 + 2p)$, so that we can take the limit and conclude

$$\hat{A}|\Gamma, p_l\rangle = \frac{1}{2}\, l_P^2 \sum_l \sqrt{p^2 + 2p}\; |\Gamma, p_l\rangle \tag{49}$$

where the sum ranges over all links l crossing the surface. The spectrum given in eq.(2) follows immediately.[8]

This result admits a surprisingly simple interpretation. Recall that a rope l of color p_l corresponds to a parallel propagator (in the connection representation) in the $j_l = p_l/2$ representation, and take units in which the Newton constant G is one, so that $l_P^2 = \hbar$. Then we can write the area of

[8]This formula corrects the numerical factor reported in [3], which was incorrect due to a miscounting of trace factors.

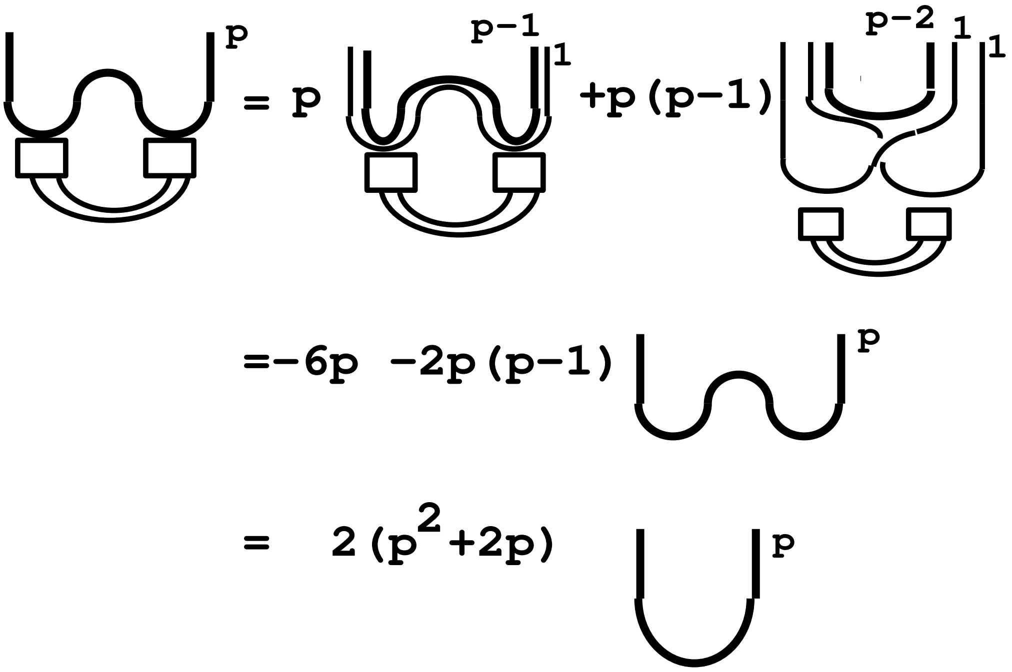

Figure 13: The calculation of the action of the area operator

the surface as

$$\mathcal{A} = \sum_l \hbar\, \sqrt{j_l(j_l+1)}. \tag{50}$$

Since the $\hbar\,\sqrt{j(j+1)}$ is the angular momentum L associated to the j-th representation of $SU(2)$, we have that,

$$\mathcal{A} = \sum_l L_n = L_{\text{total}} \tag{51}$$

where L_n is the angular momentum of the $SU(2)$ representations associated with each rope crossing the surface. See reference [24]. This remarkable fact has been pointed out by J. Iwasaki [25].

5 Spectra as physical predictions

Can we relate the spectra that we have computed to physical measurements of real volumes and areas? If we want to measure the volume of a region

R, we have to physically identify, or specify, the region R in some way or another. If we disregard the relativistic properties of the gravitational field, a region R can be identified by means of physical objects, and we can assume that these objects are dynamically decoupled from the gravitational field. As is well known, this procedure cannot be extended to the relativistic gravitational theory: there is no physical object whose dynamics is decoupled from gravity, nor, actually, is there any meaning to such a decoupling. In a relativistic gravitational context we must regard any object used to identify a physical region as dynamically coupled to the gravitational field [12]. Therefore the volume and area that we measure are always determined by objects (in a field theoretical perspective, fields) which are components of the dynamical theory.

The remark, which we believe represents the heart of the physical meaning of general covariance[9], is reflected in the mathematics of the theory as follows. The volume and area observables V and $\mathcal{A}$ considered in the previous sections are defined in pure gravity and thus are functionals of the sole gravitational degrees of freedom: $V[g], \mathcal{A}[g]$. A such, they do not commute with the canonical constraints, that is, they do not transform as singlets under 4-dimensional diffeomorphisms of the dynamical variables on which they depend (g); therefore they are not physical observables of the theory. However, if matter fields ϕ are coupled to the theory, and if we consider volumes V_{Ph} and areas $\mathcal{A}_{Ph}$ of regions *specified by those matter fields,* then such volumes and areas do commute with the constraints. Indeed $V_{Ph} = V_{Ph}[g, \phi]$ and $\mathcal{A}_{Ph} = \mathcal{A}_{Ph}[g, \phi]$ are functionals of the gravitational field as well as of the matter fields, which determine the domain of integration; a diffeomorphism of all the dynamical variables generated by the constraints acts then on the integrand as well as the integration domain. See [12, 13, 14, 15] for examples. The observables V_{Ph} and $\mathcal{A}_{Ph}$ that express volumes and areas of regions physically identified by dynamical matter are the quantities in the theory that truly correspond to the volumes and areas that we concretely measure. Our problem is therefore to understand the relation between the spectra of V and $\mathcal{A}$ computed above and the spectra of V_{Ph} and $\mathcal{A}_{Ph}$. We are now going to argue that the second ones coincide with the first ones.

We may begin this discussion by noting that we can always fix (perhaps partially) the diffeomorphism gauge invariance by constraining the values of certain matter fields at given spacetime coordinates or, equivalently, by defining preferred spacetime coordinates in terms of these matter fields. In

[9] or, equivalently, diffeomorphism invariance.

the gauge fixed theory, the matter degrees of freedom that are employed in the gauge fixing disappear, but the gauge freedom is also restricted, so that the total number of degrees of freedom of the theory remains the same. Let us sketch two examples.

First, consider a partial gauge fixing. In order for the volume of the universe (assuming this is spatially compact) to have any meaning at all in the general case in which there are no symmetries, we have to specify a synchronization of clocks; so that a spacelike surface, having a volume, is singled out. However we achieve this concretely, we can formally represent the field of clocks by means of a scalar field $\phi(\vec{x}, t)$, whose dynamics reproduces the evolution of the clocks in coordinate time. Let then R_T be the region of the points where the clocks indicate T, namely where the scalar field has value T. The volume of R_T is given by

$$\begin{aligned} V_{Ph} &= \int_{R_T} \sqrt{\det {}^3g} \\ &= \int d^4x \ \delta^4(\phi(x), T) \sqrt{\det {}^4g \ {}^4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} \\ &= V_{Ph}[g, \phi], \end{aligned} \tag{52}$$

where we have indicated by ${}^3g$ the spacial metric and by ${}^4g$ the 4-dimensional one. It is easy to verify that V_{Ph} commutes with all the constraints (including the hamiltonian constraint) [12]; indeed, this is a "physical volume", in the sense defined above. Now, the point is that we can gauge fix the hamiltonian constraint by requiring that $\phi(\vec{x}, t) = t$. In this gauge we obtain a highly non-trivial hamiltonian, which depends on the actual dynamics of ϕ, and the only remaining constraints are the diffeomorphism constraint. Thus, in this gauge the theory is formally equivalent to pure GR, but with no hamiltonian constraint and a non-trivial hamiltonian. With respect to the preferred physical time coordinate that we have selected, the 3-dimensional geometry of space becomes a physical observable; notice that the 3-dimensional geometry contains 3 degrees of freedom, which are precisely the 2 degrees of freedom of the pure gravitational field plus the 1 of the scalar field, which has disappeared. Now, in the gauge fixed theory the gauge-invariant observable quantity V_{Ph} is given by

$$V_{Ph} = \int d^3x \ \sqrt{\det {}^3g}, \tag{53}$$

But this observable has precisely the form of the non-gauge-invariant volume (3). This gravity+matter gauge fixed theory admits a quantization fully parallel to the pure gravity theory, only the hamiltonian constraint has now been replaced by a hamiltonian. However, nothing in the construction of sections 2 and 3 depend on how the dynamics is to be treated. Therefore the quantum operator representing V_{Ph} in the gauge-fixed theory with matter has the very same form as the one representing V in the pure gravity theory. Therefore the two have the same spectrum.

As a second example, consider the case in which we model a full material reference system by means of four scalar matter fields $\phi^\mu(x)$, with some fixed dynamics, and we gauge fix the theory entirely by defining coordinates $x^\mu = \phi^\mu(x)$. The case in which such a matter describes dust has been recently analyzed in great detail by Brown and Kuchař [17] from the point of view of canonical quantization. In the fully gauge fixed theory there is no remaining constraints (there is a highly non-trivial hamiltonian). The full gravitational field, as a function of the preferred coordinates, is observable. If we now fix a spacial region R in the (preferred) coordinate space, the volume of any region R is observable. It represents the volume of a region identified by material objects described by the $\phi^\mu(x)$ fields. Again, in the gauge fixed theory the operator that corresponds to the volume of R is precisely the volume operator studied in the paper.

At first sight, the result of the above discussion has a counter-intuitive aspect. Indeed, one may be tempted to object that while quantum properties of gravity can yield the quantization of, say, the volume of a region R, we must take into account the possibility that the matter identifying R must also be quantized and thus subject to quantum fluctuations. Thus, the discrete spectra found here might be smeared out into a continuum by the quantum fluctuations of the matter fields by which the region R is defined.

However, a more careful analysis shows that this consideration is wrong. The reason is that it neglects diffeomorphism invariance. Indeed, the mistake is to assume that the matter that identifies the region *as well as* all of the components of the metric tensor, which determines the volume of the region, independently undergo quantum fluctuations. This is wrong because all these component fields, being not gauge invariant, cannot represent independent degrees of freedom of the quantum theory. Instead, only the gauge invariant quantities that represent true dynamical degrees of freedom are subject to the quantum uncertainty principle and to quantum fluctuations. Therefore, in a general covariant theory, the kinematical volume by itself is not a gauge invariant degree of freedom, nor is the value of the matter fields in a coordinate region. The dynamical variable that expresses the volume is a combination of the two, and it is only this combination which is dynamically meaningful and which is quantized. One can pick a gauge by fixing the coordinates to the values of the matter fields, in which case it appears that the dynamical quantum degrees of freedom are represented by certain components of the metric. Or one could fix components of the metric, so that the matter fields represent, in such a gauge, the dynamical degrees of freedom. But these are equivalent descriptions of the same physics. What we cannot do is pick a gauge which allows us to view both the metric and the reference matter as independent quantum objects. To put it differently, reference matter is not quantized, in a gauge in which it is used up by the gauge fixing, for the same reason for which the Higgs field is not treated as a quantum field in those gauges in which its degree of freedom is represented by the longitudinal components of the gauge bosons.

In conclusion, since the calculations of sections 2 and 3 do not depend on the dynamics determined by the hamiltonian constraint, we can reinterpret these calculations as referring to physical areas and volumes determined by reference matter. Therefore, as we have emphasized in the introduction, we may conclude that the the spectra of volume and area computed here may be considered to constitute physical predictions of the quantum theory of gravity in the loop representation.

6 Conclusions

Many different approaches to quantum gravity incorporate a fundamental length. Well-known semiclassical arguments, some simple, other more sophisticated, suggest that geometrical measurements must be limited by an intrinsic quantum uncertainty of the order of the Planck length, in quantum general relativity [26], as well as in string theory [27]. In string theory, in particular, the existence of a short scale discrete structure has been indicated as a possible reason for the exponential damping of high energy scattering amplitudes [28] and for the slower growth of the thermal partition function at high temperature [29], and the intriguing short scale string discreteness discovered by Klebanov and Susskind [30] has been recently indicated by Horowitz as a suggestive indication of closeness between string theory and loop representation [31]. (On the relation between string theory and loop representation, see [32].) A relation between black hole entropy and area quantization was suggested by Bekenstein [33] and recently studied by Jacobson [34], and some kind of short scale discreteness is implicit in the idea of spacetime foam [35]. For an interesting and comprehensive overview of many of these ideas on a fundamental length, see the recent review [36], which concludes noticing that "the presence of a lower bound to the uncertainty of geometry measurements seems to be a model-independent feature of quantum gravity".

Here, such a lower bound, and its associated spectral discreteness, derive from the straightforward non-perturbative quantization of general relativity. The discreteness is specific, as we have detailed, non-trivial formulae for the spectra of volume and area (equations (1) and (2)).

A characteristic aspect of our results is that they are entirely kinematical. As the hamiltonian constraint plays no role in the derivation, the results must be independent of the dynamics of general relativity. In this respect, what we have done here may be considered to have carried quantum gravity to the same point that normal ordering carried quantum electrodynamics. In that case, the correct treatment of a kinematical operator product divergence results in the discovery of the physical content of the theory: photons, electrons and positrons. The stage is set for the definition of dynamics via the hamiltonian. Here, a more complicated procedure, made necessary by the requirement of diffeomorphism invariance and the absense of a background metric and an associated positive/negative frequency distinction, reveals the physical content of the theory: the discrete states labeled by the spin networks. Again, what remains is to apply the dynamics to these states, either via the hamiltonian constrant or an appropriately defined hamiltonian [4].

The fact that the results found here derive only from a succesful treatement of kinematical, operator product, divergences, gives them a certain robustness. To the extent that the results here stand as predictions of quantum general relativity, they are also predictions of quantum supergravity, or any of the large number of modifications of general relativity such as those that involve dilaton fields or higher derivative terms.

The discreteness of areas and volumes found here then supports and strengthens the evidence for the existence of a discrete short-scale structure of space, which emerged first from studies of the kinematical "weave" states [3, 37]. However, these results go further in extending these results to the diffeomorphism invariant level, and in providing a precise physical meaning to the claim of the existence of a short-scale structure, one tied to the topological discreteness of knot space. This might have far reaching consequences for our understanding of the general structure of infinite-dimensional, diffeomorphism invariant quantum field theories. Indeed, notice that the framework here is substantially dissimilar from the conventional framework in which discreteness of the regularized theory is scaled away by the large scale limit taken at a critical point that defines a continuum theory. Here the discreteness is physical. As a result, the relation between short distance and long distance regimes of the theory is very different than in quantum field theories defined on metric manifolds. The continuum limit must be a limit in which states admit regions of large volume (on the Planck scale), and behave semiclassically, not in which a short distance cutoff is taken to zero.

This does not mean that further divergences might yet be encountered in the evaluation of the dynamics of the theory. However, as the handling of the kinematical divergences has, in this case, revealed a state space with a natural discrete structure and short distance cutoff, we do not expect anymore the appearance of the conventional ultraviolet divergences, which come from summing over the free Fock states of arbitrarily short wavelength. Here the possible sources of divergences correspond to infinitely large networks, or to arbitrarily large values of the labels on the graphs. But, by the results found here, these correspond to limits of large volumes and large areas; hence these must be infrared divergences.

There remains still much to be understood about these issues. In particular, it is important to note that the Planck constant l_P appearing in the spectra (1) and (2) is a bare quantity, that might suffer a finite renormalizations and thus not coincide with the macroscopic value of $\sqrt{\hbar G_{Newton}/c^3}$.[10]

Finally, we must mention that we can make no claim that the spectra computed here are unavoidable consequences of just quantum theory and general relativity, since choices and assumptions are made in the construction of any specific quantum theory. Assumptions about the nature of the state space on which the theory is constructed may affect the theory substantially. Ashtekar and others [6, 7] are developing a deep mathematical analysis of the loop representation capable of addressing these issues. A case to be examined in this regard is the extended loop representation [38]: it will be interesting to see if the present results can be derived in that case as well. On the other hand, the spectra computed here make the present form of the loop representation theory falsifiable, in spite of its incompleteness.

In the absence of Planck scale measurements, it is of course hard to imagine how predictions for the discreteness of these observables could be tested. However, the examples of the early developments of quantum mechanics and solid state physics suggests that this discreteness could have implications for the thermodynamics of the gravitational field [39]. This could have impor-

[10] For an indication that this does occur, see [4].

tant implications for the interpretation of black hole entropy, as well as for the question of the production, in the very early universe, of a spectrum of primordial gravitational radiation. These questions are presently under investigation.

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