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Dynamics of Systems of Two Close Planets

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In connection with the study of newly formed protoplanetary embryos in the early Solar System, we study the dynamics of a pair of interacting planets orbiting a Sun. By examining the topological stability of the three-body problem, one finds that for initially circular planetary orbits the system will be Hill stable (that is, stable against close approaches for all time) if the fractional orbital separation $\Delta > 2.4(\mu_1 + \mu_2)^{1/3}$, where μ_1 and μ_2 are the mass ratios of the two planets to the Sun. The validity of this stability condition is supported by numerical integrations. The chaotic dynamics of these systems is investigated. A region of bound chaos exterior to the Hill-stable zone is demonstrated. The implications for planetary accretion, the current Solar System, and the pulsar planet system PSR 1257 + 12, are discussed. © 1993 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to attempt to understand the qualitative features (and make some quantitative predictions) of the dynamics of a coplanar system with a large central mass (a "Sun") and two smaller masses ("planets"). This is of relevance to the question of the stability of the Solar System. One wishes to discover the conditions under which the planets will forever remain in orbits with low eccentricities and roughly their current semimajor axes. The historical approach has been to apply secular perturbation theory to Lagrange's planetary equations (see Milani 1988 for a review). However, this theory predicts that, to second order in the masses, the semimajor axes of the planets have no secular growth, which would seem to imply that in principle the system must be stable forever. Being more careful, one discovers that the Lagrange solution contains terms with the infamous small divisors at higher order; accordingly, in general the neglected higher-order terms can become important provided that the frequencies of these terms are slow enough. Indeed, we will see that if two planets are in orbits of close enough proximity then the semimajor axis variation appears immediately in a secular fashion; thus the secular theory, which relies on time averaging, is useless in predicting the behavior of the system.

This classical problem has gained renewed interest due to the discovery of two planets around the pulsar PSR1257 + 12. The stability analysis performed below will allow an upper limit to be placed on the mass of the planets (see Rasio *et al.* 1992).

This problem is also of interest because of its relevance to planetary formation. In currently popular theories of planetary formation relatively large embryos are produced during a phase of rapid "runaway growth" (see Safronov 1991 for a review). As a consequence there are likely to be several large protoplanetary embryos on nearly circular orbits. Each embryo will have swept up almost all the material in its "feeding zone," given by numerical simulation as roughly $\sim 3.5\,R_H$, where R_H is the radius of its Hill sphere (Lissauer 1987). In the terrestrial planet region such embryos would have a mass of $\sim 10^{25}$ g $\sim 10^{-8}\,M_\odot$. In the outer planet region the embryos may be the so-called "super-ganymedian puffballs" which would contain several Earth masses of material (i.e., $\sim 10^{-5}$ $M_\odot$). What is the minimum spacing of such embryos such that they would be stable against close approach and thus would be the largest objects formed (locally) in the disk? The answer is partially provided by the solution of the simplified problem discussed below.

Two different notions of stability are of interest. A system is "Hill stable" (or just "stable") if the planets are forbidden to undergo close approaches for all time (a close approach is defined in Section 3). A system is "Lagrange stable" if in addition the semimajor axes of the planets are bounded for all time. We shall see below that it is possible to prove Hill stability for some two-planet systems, but we cannot prove them to be stable in the sense of Lagrange.

The following notation is used. Without loss of generality, choose units so that the semimajor axis of the inner planet is unity and its (osculating) orbital period is 2π.

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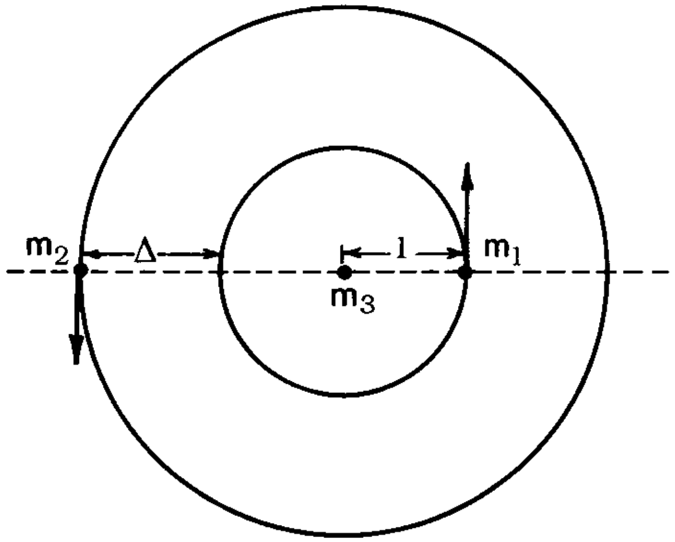

FIG. 1. The model problem. A full three-body problem with the restriction that $m_1 + m_2 \ll m_3$. The initial osculating semimajor axis of m_1 is taken to be unity while that of m_2 is $1 + \Delta$. The figure shows the planets beginning with an initial longitude difference $\delta f = 180°$. One then asks: what is the minimum value of Δ required for stability? The orbits need not be circular (nor coplanar).

The initial semimajor axis of the outer planet is denoted by $a = 1 + \Delta$, so that Δ is the initial orbital separation. The masses of the inner planet, outer planet, and Sun are (respectively) m_1 m_2, and m_3 with $m_1 + m_2 + m_3 \equiv 1$ in our units, with $m_1 + m_2 \ll m_3$. The unit of time is chosen so that $G = 1$. Although Fig. 1 shows initially circular orbits this is not a constraint in what follows. The question we seek to answer is: what is the minimum Δ such that the system is Hill stable?

Section 2 discusses analytic stability criteria, including a development of the topological stability criterion for the three-body problem. Section 3 presents numerical experiments that confirm the analytical results and explore the chaotic nature of the problem. Section 4 discusses the relevance of the results for planetary formation, the current Solar System, and the pulsar system PSR 1257+12. Section 5 presents the conclusions.

2. ANALYTIC STABILITY RESULTS

2.1. *Circular Restricted Three-Body Problem*

If we consider the case $m_2 \rightarrow 0$ and the inner planet on a circular orbit (with eccentricity $e_1 = 0$) then we recover the circular restricted three-body problem. We can establish Hill stability for the system through the use of the well-known Hill surfaces, also called the zero-velocity surfaces (see Danby 1988). Since such surfaces cannot be crossed by the test particle, their presence between the test particle and the planet forbids the test particle from ever approaching the planet. In this way one can prove Hill stability for large enough initial separation Δ. For test particles on initially circular orbits with an initial longitude difference δf with respect to the planet (of mass ration $\mu_1 \equiv m_1/m_3$) these critical separations are:

$$\Delta > 2.4\,\mu_1^{1/3} \quad \text{when } \delta f = 180° \tag{1}$$

and

$$\Delta > 2.1\,\mu_1^{1/3} \quad \text{when } \delta f = 0°. \tag{2}$$

(see Birn 1973, Gladman and Duncan 1990). These results should be recoverable as limits in any generalized criterion.

2.2. *Hill's Problem*

The stability of nearly coorbital planets can be approached by an examination of Hill's problem. The notation of the excellent paper by Henon and Petit (1986) will be used (excepting a renumbering of the bodies). When the planets are far apart, their relative motion (expressed in relative Hill's coordinates which scale as the one-third power of the mass ratios) is given by

$$\xi = D_1 \cos t + D_2 \sin t + D_3 \tag{3}$$

$$\eta = -2D_1 \sin t + 2D_2 \cos t - \frac{3}{2} D_3 t + D_4, \tag{4}$$

where ξ is the radial and η the azimuthal coordinate. The D_j are constants that are related to the orbital elements of the planets by

$$a_i = a_o(1 + m^{1/3} D_{i3}), \quad e_i = m^{1/3}\sqrt{D_{i1}^2 + D_{i2}^2} \quad (i = 1, 2) \tag{5}$$

and

$$D_j \equiv D_{2j} - D_{1j} \quad (j = 1, 2, 3, 4); \tag{6}$$

$m \equiv m_1 + m_2$, a_o is the radius of the mean orbit, and D_4 can be set to zero by rescaling the time variable. In these coordinates a second integral, analogous to the Jacobi integral of the restricted problem, is given by

$$3\xi^2 + \frac{2}{\sqrt{\xi^2 + \eta^2}} - \dot{\xi}^2 - \dot{\eta}^2 \equiv \Gamma. \tag{7}$$

For $\Gamma > 3^{4/3}$ this integral divides the regions of allowed motion into three separate parts (Henon 1970). We shall use this to our advantage. For an initially asymptotically circular encounter with an orbital separation Δ_H (in Hill's units), one can write

$$\xi = \Delta_H, \quad \eta = -\frac{3}{2} t \Delta_H, \quad \dot{\xi} = 0, \quad \dot{\eta} = -\frac{3}{2}\Delta_H. \tag{8}$$

The planets will be separated from each other by a forbidden surface if

$$\Gamma = 3\Delta_H^2 - \frac{9}{4}\Delta_H^2 > 3^{4/3}, \tag{9}$$

neglecting the second term in Eq. (7) for large initial separation. This demands, in our units, that

$$\Delta = \Delta_H m^{1/3} > 2 \cdot 3^{1/6}(m_1 + m_2)^{1/3} \simeq 2.40(m_1 + m_2)^{1/3}, \tag{10}$$

which reduces to Eq. (1) when $m_2 \rightarrow 0$. Ida and Makino (1993) give a similar approach that includes orbital inclinations.

The above analysis is strictly valid only for one encounter since Hill's equations are a truncated set that just describe the motion while the planets are near each other. During the encounter the constants D_{ij} of the planets change due to the mutual perturbation, but the outgoing value of the Jacobi constant

$$\Gamma = \frac{3}{4}D_3^2 - (D_1^2 + D_2^2) \tag{11}$$

must be conserved. Since D_3 is related to the semimajor axis separation of the planets (Eq. (5)), and $(D_1^2 + D_2^2) = (ae)^2$ to lowest order (see Duncan *et al.* 1989), these quantities will be conserved in the Keplerian motion that brings the bodies to their next conjunction. Thus it is possible to "patch" the encounters together, and the conserved Jacobi integral thus prevents the planets from undergoing a close approach for all time.

However, this analysis has a drawback once we attempt to apply the analysis to the motion away from conjunction. One is unsure at what level the approximations inherent in Hill's equations and the requirement that the eccentricities are small may affect the results. It is not completely clear that the patching required from conjunction to conjunction will not introduce small errors that will invalidate a stability result that is meant to be valid for all time. The following analysis escapes these uncertainties.

2.3. Topological Stability

In the 1970s considerable progress was made in understanding the three-body problem when it was shown that a bifurcation existed in the dynamics. In particular one could prove that for certain initial conditions there existed a division of the phase space into regions of allowed and disallowed motion. Originally a result from algebraic topology, this result was introduced to celestial mechanics by many workers (see Roy *et al.* 1984, Milani and Nobili 1983b, Marchal and Bozis 1982, Zare 1977, and references therein). In essence, the result is: the integral product c^2h (where c and h are, respectively, the total angular momentum and energy of the three-body system) is of fundamental importance to the dynamics of the system. Given only the masses of three bodies, one can compute a critical value of $(c^2h)_{\text{crit}}$ at which point the dynamics bifurcate. If the initial conditions of the system produce a value of c^2h which satisfies $c^2h > (c^2h)_{\text{crit}}$ (which, being composed of integrals of the motion, will remain satisfied for all time), then the system is Hill stable. What happens is very analogous to the circular restricted three-body problem; a surface of zero velocity forms which prevents close approaches *for all time*. However, in the general case one cannot conveniently express the location of this surface in terms of the coordinates (since the surface moves in coordinate space).

The analysis below applies the c^2h criterion to the case in which two of the bodies (the planets) have much smaller masses than the third. This study will improve on previous results by producing explicit scaling laws and by expressed the stability conditions in terms of simple expressions that involve the orbital elements of the planets.

I have adopted the notation and spirit of Marchal and Bozis (1982). They parameterize the dynamics by the ratio

$$\frac{p}{a} = -\frac{2M}{G^2 M_*^3} c^2 h, \tag{12}$$

where M and G are the total mass of the system and the gravitational constant, respectively, both equal to unity in our units, and $M_* \equiv m_1m_2 + m_1m_3 + m_2m_3$. Although not important for our purposes, p and a are respectively the "generalized" semilatus rectum and generalized semimajor axis of the three-body system. Note the presence of the parameter c^2h.

Marchal and Bozis (1982) give the critical value at which the bifurcation occurs in the phase space topology as

$$\left(\frac{p}{a}\right)_{\text{crit}} = 1 + 3^{4/3}\frac{m_1m_2}{m_3^{2/3}(m_1 + m_2)^{4/3}} - \frac{m_1m_2(11m_1 + 7m_2)}{3m_3(m_1 + m_2)^2} + \dots, \tag{13}$$

if $m_1 > m_2$ (if $m_2 > m_1$ just interchange their subscripts). For a system with ratio p/a larger than this critical value the planets are forbidden from having a close approach (i.e., the system is Hill stable). This is a very straightforward condition in principle since the critical value depends on only the masses and, for any set of initial conditions, we can compute the ratio p/a. No approximations have been invoked (beyond the fact that the value of the parameter at the bifurcation is given in a power series, but may be computed numerically to any accuracy). Note that there is no restriction to a coplanar system required for this condition, although below we assume this to be the case. The inclusion of inclinations will not change the application fo the criterion, and the forbidden surfaces extend out of the plane as in the circular restricted three-body problem. Note that this analytic result provides a sufficient, but not necessary condition for stability. That is, *nothing can be said from the analytical theory* about a system failing to satisfy the topological criterion. This will be discussed further below.

Now compute the stability condition for the two-planet case. Since the total mass of the system is unity and the masses of the planets are very small, the masses of the planets will be denoted by their mass ratios with the central body:

$$\mu_1 \equiv \frac{m_1}{m_3} \simeq m_1 \ll 1, \quad \mu_2 \equiv \frac{m_2}{m_3} \simeq m_2 \ll 1. \tag{14}$$

The angular momentum and energy of the system can easily be computed from the initial conditions in terms of the osculating orbital elements of the planets. If we use the system barycenter as the origin, then to lowest order

$$\begin{aligned} c &= \sum_{i=1}^{3} m_i(\mathbf{r}_i \times \mathbf{v}_i) \\ &= \mu_1\sqrt{a_1(1-e_1^2)} + \mu_2\sqrt{a_2(1-e_2^2)} + \cdots \\ &\simeq \mu_1\sqrt{1-e_1^2} + \mu_2\sqrt{(1+\Delta)(1-e_2^2)}, \end{aligned} \tag{15}$$

where we have used $a_1 = 1$ and $a_2 = 1 + \Delta$. The contribution to the angular momentum from the central mass is of order of the square of the mass ratio of the larger planet and has thus been neglected. We assume that the initial conditions place the planets farther apart than their Hill spheres, and thus the energy of the system can be written

$$\begin{aligned} h &= -\frac{\mu_1}{2a_1} - \frac{\mu_2}{2a_2} + \cdots \\ &\simeq -\frac{\mu_1}{2} - \frac{\mu_2}{2(1+\Delta)}. \end{aligned} \tag{16}$$

Again, the contribution of the central object is negligible. For notational ease, let us define the commonly occurring quantities

$$\gamma_i \equiv \sqrt{1-e_i^2}, \tag{17}$$

$$\alpha \equiv \mu_1 + \mu_2, \tag{18}$$

and

$$\delta \equiv \sqrt{1+\Delta}. \tag{19}$$

Thus δ is now our measure of the separation of the orbits. Then for our system we can compute (using Eq. (12))

$$\frac{p}{a} = (\mu_1 + \mu_2)^{-3}\left(\mu_1 + \frac{\mu_2}{\delta^2}\right)(\mu_1\gamma_1 + \mu_2\gamma_2\delta)^2. \tag{20}$$

The system will thus be Hill stable if

$$\alpha^{-3}\left(\mu_1 + \frac{\mu_2}{\delta^2}\right)(\mu_1\gamma_1 + \mu_2\gamma_2\delta)^2 > 1 + 3^{4/3}\frac{\mu_1\mu_2}{\alpha^{4/3}} \tag{21}$$

to lowest order. Clearly for large enough δ (and hence large enough orbital separation Δ) the inequality is satisfied and the system will be Hill stable (assuming $\gamma_2 \neq 0$, which would correspond to $e_2 = 1$). Note that any limits in which either mass vanishes should be taken later, since at this stage it makes the problem indeterminate. Multiplying through by δ^2 will yield a quartic equation for the critical value of δ_0 such that $\delta > \delta_0$ means that Hill stability can be proven. This equation is usually solved numerically in the literature. However, we can extract convenient analytical scaling laws in several subcases.

Initially circular orbits. For zero eccentricity orbits ($\gamma_1 = \gamma_2 = 1$) the quartic equation for the critical value of δ is

$$\begin{aligned} \mu_1\mu_2^2\delta_0^4 + 2\mu_1^2\mu_2\delta_0^3 & \\ + \delta_0^2(\mu_1^3 + \mu_2^3 - \alpha^3 - 3^{4/3}\mu_1\mu_2\alpha^{5/3}) & \\ + 2\mu_1\mu_2^2\delta_0 + \mu_1^2\mu_2 &= 0. \end{aligned} \tag{22}$$

Since the mass ratios are small we surmise that the critical separation Δ_c will be also and expand all powers of δ_0 to second order in Δ_c. For example, $\delta_0 = \sqrt{1+\Delta_c} \simeq 1 + \Delta_c/2 - \Delta_c^2/8$, etc. One then finds that

$$\begin{aligned} \Delta_c &= 2 \cdot 3^{1/6}(\mu_1 + \mu_2)^{1/3} \\ &+ \left[2 \cdot 3^{1/3}(\mu_1 + \mu_2)^{2/3} - \frac{11\mu_1 + 7\mu_2}{3^{11/6}(\mu_1 + \mu_2)^{1/3}}\right] + \cdots \end{aligned} \tag{23}$$

or, to lowest order in the masses,

$$\Delta_c \simeq 2.40\,(\mu_1 + \mu_2)^{1/3}. \tag{24}$$

The higher-order terms were computed using the full expression for the bifurcation value in Eq. (13) and the computer algebra package MACSYMA. Note that this result reduces to the correct restricted limit as $\mu_2 \to 0$

(Eq. (1)). Also note that this produces the convenient scaling for equal mass planets ($\mu_1 = \mu_2 = \mu$) of

$$\Delta_c \simeq 3\mu^{1/3}. \tag{25}$$

Equal masses, small eccentricities. With $\mu_1 = \mu_2 = \mu$ the quartic equation becomes

$$\gamma_2^2\delta_0^4 + 2\gamma_1^2\gamma_2\delta_0^3 - \delta_0^2(8 - \gamma_1^2 - \gamma_2^2 + 8\beta\mu^{2/3}) + 2\gamma_1\gamma_2\delta_0 + \gamma_1^2 = 0. \tag{26}$$

with the abbreviation $\beta \equiv (3/2)^{4/3}$. For small initial eccentricities ($e \lesssim \mu^{1/3}$) we write $\gamma_i \simeq 1 - e_i^2/2$ and then expand in the e_i and in Δ_c as before to obtain

$$\Delta_c = \frac{2\sqrt{6}(e_1^2 + e_2^2 + 2\beta\mu^{2/3})^{1/2}}{3} + \frac{16\beta^2\mu^{4/3} - 3\sqrt{6}\mu\sqrt{e_1^2 + e_2^2 + 2\beta\mu^{2/3}} + (14e_1^2 + 18e_2^2)\beta\mu^{2/3} + 3e_1^4 + 8e_1^2e_2^2 + 5e_2^4}{3(e_1^2 + e_2^2 + 2\beta\mu^{2/3})} + O(\mu), \tag{27}$$

which to lowest order is

$$\Delta_c \simeq \sqrt{\frac{8}{3}(e_1^2 + e_2^2) + 9\mu^{2/3}}. \tag{28}$$

This reduces to our previous equal mass case result (Eq. (25)) for zero eccentricities. Notice that the higher-order terms exhibit an asymmetry in e_1 and e_2 that one would expect (since the problem should not be completely symmetric with respect to the two planets).

Equal masses, equal but large eccentricities. For the case of equal and arbitrary large eccentricities one can also produce an analytic expression for the minimum Δ necessary for Hill stability. If one sets $\gamma_1 = \gamma_2 = \gamma$, we can solve for δ_0 in the full quartic (Eq. (26)), pick out the root larger than unity, and solve for Δ_c using computer algebra. Since we are concerned here with large eccentricities, we can neglect the mass terms (of order $\mu^{1/3}$) compared to γ and find that

$$\Delta_c = \left(\sqrt{\frac{3 + e^2}{2(1 - e^2)} - \frac{1}{2}\sqrt{\frac{9 - e^2}{1 - e^2}}} + \frac{1}{2}\sqrt{\frac{9 - e^2}{1 - e^2}} - \frac{1}{2}\right)^2 - 1 + \ldots, \tag{29}$$

where the neglected terms are of higher order in the mass ratios. The required semimajor axis separation as a function of eccentricity of the planets is plotted in Fig. 2 (see also Donnison and Williams 1983). If we expand in a Taylor series about $e = 0$ for small e, then

$$\Delta \sim \frac{4}{\sqrt{3}}e + \frac{8}{3}e^2 + \ldots, \tag{30}$$

as we would expect from Eq. (28) (but note that then we have no right to ignore the mass terms). Note that this last condition insures that the orbits are not crossing. In our units the aphelion of the inner planet is at $Q = 1 + e$ and the perihelion of the outer planet at $P = (1 + \Delta)(1 - e)$ and their (unperturbed) distance of closest approach is

$$P - Q = \left(\frac{4}{\sqrt{3}} - 2\right)e \simeq 0.3e > 0.$$

3. NUMERICAL STUDIES

Numerical simulations of the three-body problem were used to explore the validity (and usefulness) of the expressions derived above and to examine the sometimes chaotic nature of the solutions. One must acknowledge the usual

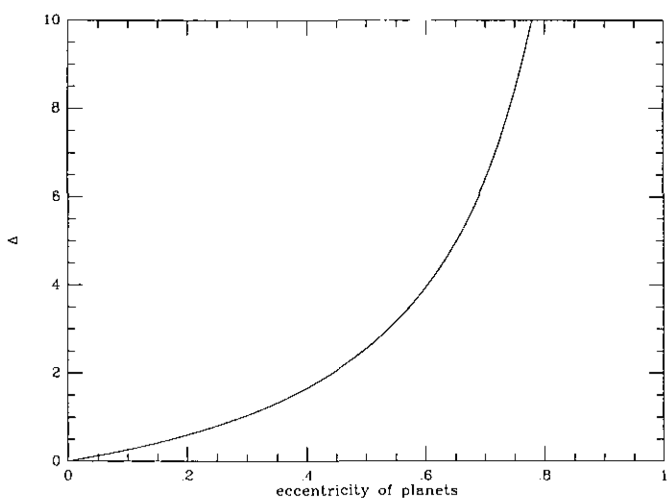

FIG. 2. The required orbital separation for two equal-mass planets as a function of their initially equal eccentricity (Eq. (29)). This result is invalid once the eccentricity becomes small ($\lesssim \mu^{1/3}$) at which point Eq. (27) should be used.

TABLE I
Results of Simulations of Systems with Initially Circular Orbits and Masses $\mu_1 = \mu_2 = 10^{-5}$

Δ	N_c
0.050	2
0.060	11
0.061	10
0.062	81
0.063	5
0.064	125
0.065	>10,000
0.066	>10,000
0.067	>10,000
0.069	>10,000
0.070	>10,000
0.071	>10,000
0.075	>10,000
0.080	>10,000
0.085	>10,000
0.090	>10,000

Note. N_c is the number of conjunctions that the system survives without suffering a close approach. From Eq. (25) the system is Hill stable once $\Delta > 0.0646$.

proviso that we cannot prove stability through numerical simulation, although we can demonstrate instability. It is shown below that the behavior of the simulated systems matches the above predictions very well.

3.1. Integration Method

The integrations were carried out in the barycentric frame using a sixth-order symplectic integration algorithm on the full three-body equations of motion. The advantages of the symplectic methods are discussed in Gladman *et al.* (1991). The particular integrator used was supplied by Forest (1992). The simulations used a step size of 150 steps per orbit for the inner planet which kept errors in the energy and angular momentum below 1 part in 10^{11}. This step size produces a phase error growing linearly at $\sim 8 \times 10^{-10}$ radians/orbit in the corresponding Kepler problem.

The three-body systems that were integrated were always started with the planets having an initial longitude difference of 180°. If the planets had initial eccentricities they they were started at aphelion. The systems were deemed to be unstable (and the integrations terminated) if at any point the planets suffered a close approach. Close approach was defined as being closer than the size of the sphere of influence, $2\mu^{2/5}$ (Battin 1964, Roy 1988), of the more massive planet. The sphere of influence is the radius at which (from a perturbations point of view) it is better to view the motion as one planet around the other being perturbed by the "Sun" rather than the planets moving around the Sun and being perturbed by each other. Such a system is clearly not Hill stable. The results are extremely insensitive to the exact choice of termination criterion.

3.2. Initially Circular Orbits

To confirm the validity of the scaling law for initially circular planets (Eq. (24)), a series of integrations was performed. The inner planet was fixed as having mass $\mu_1 = 10^{-5}$. For various masses of the outer planet the system was integrated for a length of time that produced at least 10,000 conjunctions of the planets. One discovers that, if the initial orbital separation Δ of the planets is less than a certain value, then the planets come to close approach in usually less than 1000 conjunctions, typically less than 100 (i.e., the system is not Hill stable). For initial separations larger than this value the system never reaches close approach. Table I shows how dramatic this boundary can be.

Figure 3 shows the empirical critical separation plotted versus μ_2 and compared to the separation required for Hill stability as given by Eq. (23). Notice that once the mass of the second planet is smaller than the first by about an order of magnitude the result is basically the same as that predicted by the circular restricted problem. The sphere of influence of the inner planet in these units is only ≃0.013, so the planets are initially well separated.

The agreement here could come as a surprise. As remarked above the topological criterion provides a sufficient but not necessary condition for stability. In the case

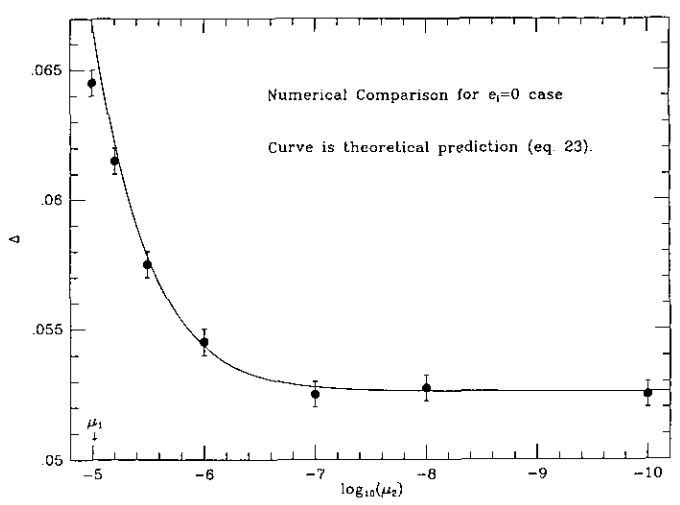

FIG. 3. Comparison of numerical stability results with the predictions of the topological stability criterion. The inner planet is given a mass ratio of $\mu_1 = 10^{-5}$. For each value of μ_2 many different values of the orbital separation Δ were investigated and the largest of these that showed Hill instability is plotted. The results agree very well with the predictions of Eq. (23), with a slight departure at the highest mass ratios where the neglected higher-order terms presumably contribute a small correction.

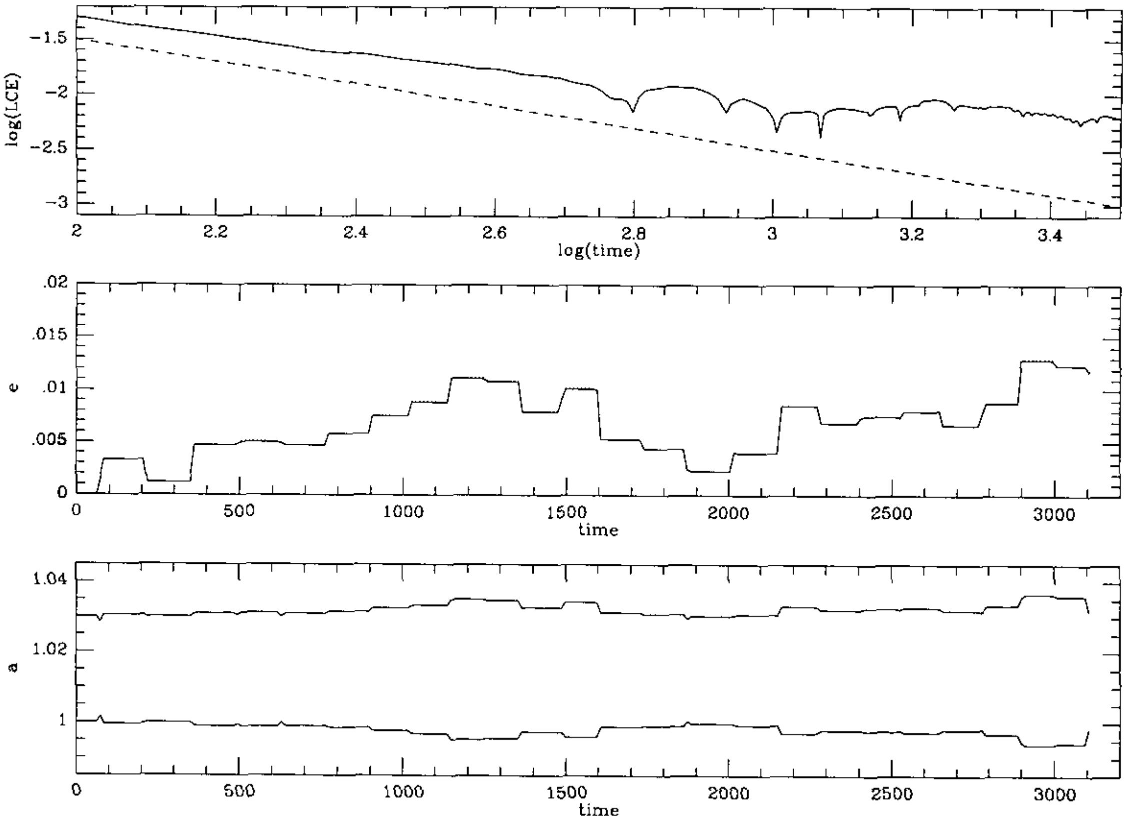

FIG. 4. Orbital history for a system with $\mu_1 = \mu_2 = 10^{-6}$, $\Delta = 0.030$, $e(t = 0) = 0$. The eccentricities of the planets are almost identical throughout the evolution. The orbital period of the inner planet is roughly 2π and the size of a planetary Hill sphere is 0.007 in these units. (Top) The Lyapunov characteristic exponent (LCE) calculation with a dashed reference line showing a (quasi-periodic) slope of -1. The Lyapunov time is $(\mathrm{LCE})^{-1} \simeq 10^{2.2} \simeq$ the synodic period.

of initially circular orbits we see that the condition is in fact necessary. That is, any system which begins with a smaller separation is seen to be unstable. This shall be discussed further below.

3.3. *Initially Eccentric Orbits*

Larger initial eccentricities require that the initial separation increase if the system is going to remain Hill stable, as predicted by Eqs. (28) or (29). However, other authors (Valsecchi *et al.* 1984, Milani and Nobili 1983a) have noted that for initially eccentric orbits the predictions of the Hill stability criterion are not as useful. That is, systems that are predicted to be Hill stable are indeed so, but systems that fail to pass the criterion may still be found to be empirically quite stable for very long periods of time. This is in contrast to the low-eccentricity case, where we have seen that failing to meet the stability requirements means that the system will come to close approach in less than 1000 conjunctions.

To illustrate this, consider the following two experiments. In the first case $\mu_1 = \mu_2 = 10^{-6}$, with $e_1 = 0$, $e_2 = 0.01$ initially. Since $e \lesssim \mu^{1/3}$ we use Eq. (28) to calculate the Hill stability separation as $\Delta = 0.034$ (compared to a value of 0.030 from Eq. (25) if the orbits were initially circular). Numerical experiments confirmed that initial separations larger than this were indeed Hill stable. However, the crossing behavior terminated at $\Delta = 0.032$, so although the nonzero eccentricity of the outer body has increased the size of the unstable zone, the topological criterion is no longer a perfect diagnostic for instability.

In the second case $\mu_1 = \mu_2 = 10^{-5}$, with $e_1 = e_2 = 0.01$ initially. Here Eq. (28) predicts $\Delta = 0.069$ (as opposed to 0.065 for initially circular orbits). However, with only one exception, regardless of whether the perihelia begin aligned or anti-aligned, the unstable zone was not increased at all over the circular case. That is, in the range $\Delta = 0.065$–0.069 the systems showed no instability. The increased stability was seen to coincide with the existence of protection mechanisms as discussed in the next section.

The topological stability criterion thus does guarantee stability if its conditions are satisfied but, for eccentric initial orbits, the system may fail to satisfy the criterion and still show empirical stability provided the orbital separation Δ is at least as large as required for stability in the circular case. The results of numerical experiments lead one to the following empirical conclusion: if the eccentricity of the larger planet is much less than (a factor of a

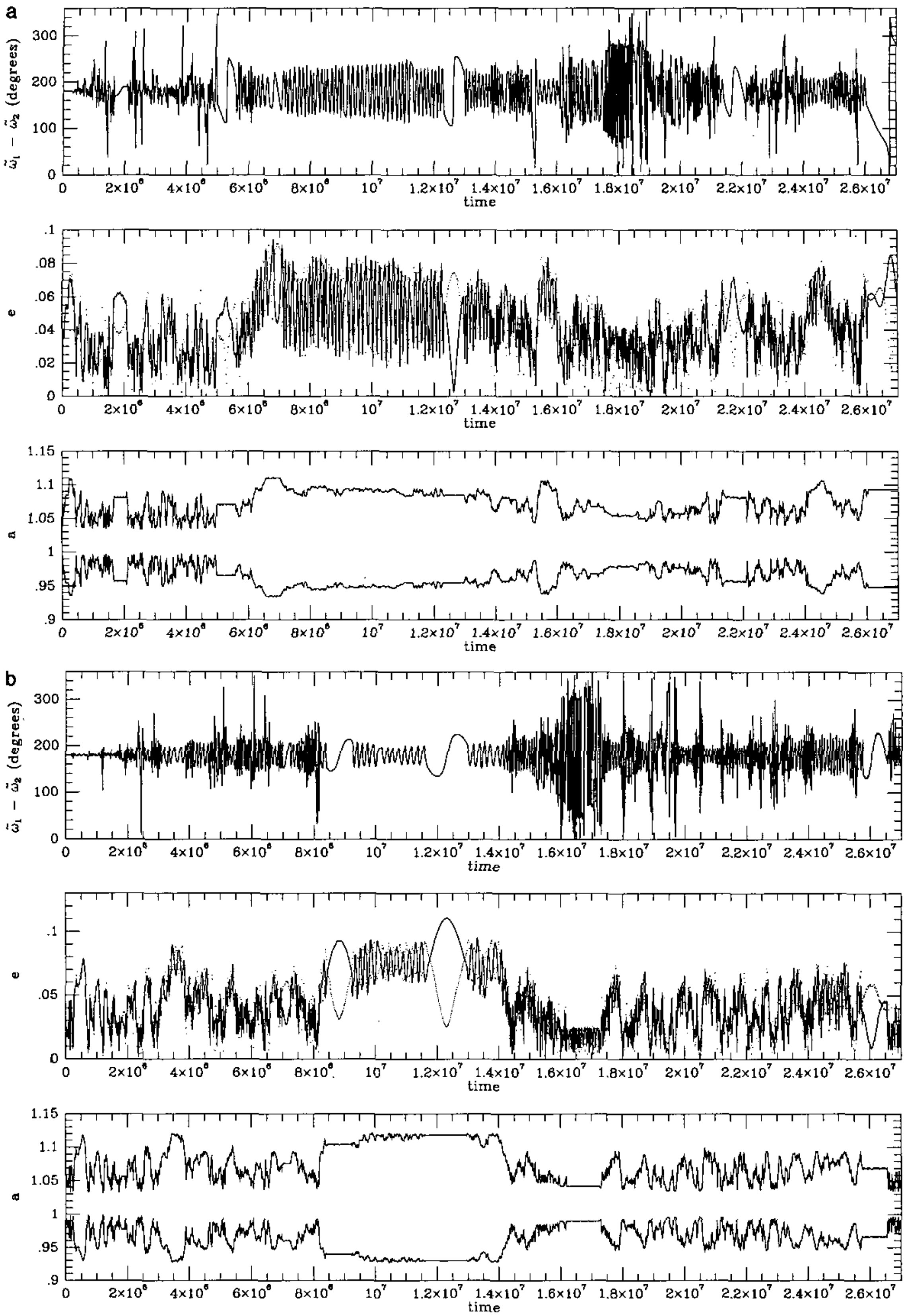

FIG. 5. These figures show osculating orbital element evolutions for the equal mass systems $\mu_1 = \mu_2 = 10^{-6}$ with initially circular orbits that start with separations just outside the crossing zone for (a) $\Delta = 0.03100$, (b) $\Delta = 0.03101$, (c) $\Delta = 0.03102$, (d) $\Delta = 0.03105$. The upper panel of each shows the difference in the perihelion longitude of the two planets. The dotted line in the eccentricity panel is for the outer planet. For all of these systems the planetary Hill spheres are roughly 0.007 and the orbital periods are about 2π. The synodic period varies as the separation varies, but for "typical" values of about $a_1 = 0.96$, $a_2 = 1.07$ (as shown in the semimajor axis panels) it is ≈ 40. Note that the average separation is quite a bit larger than the initial separation for the circular orbits. See the text for more discussion.

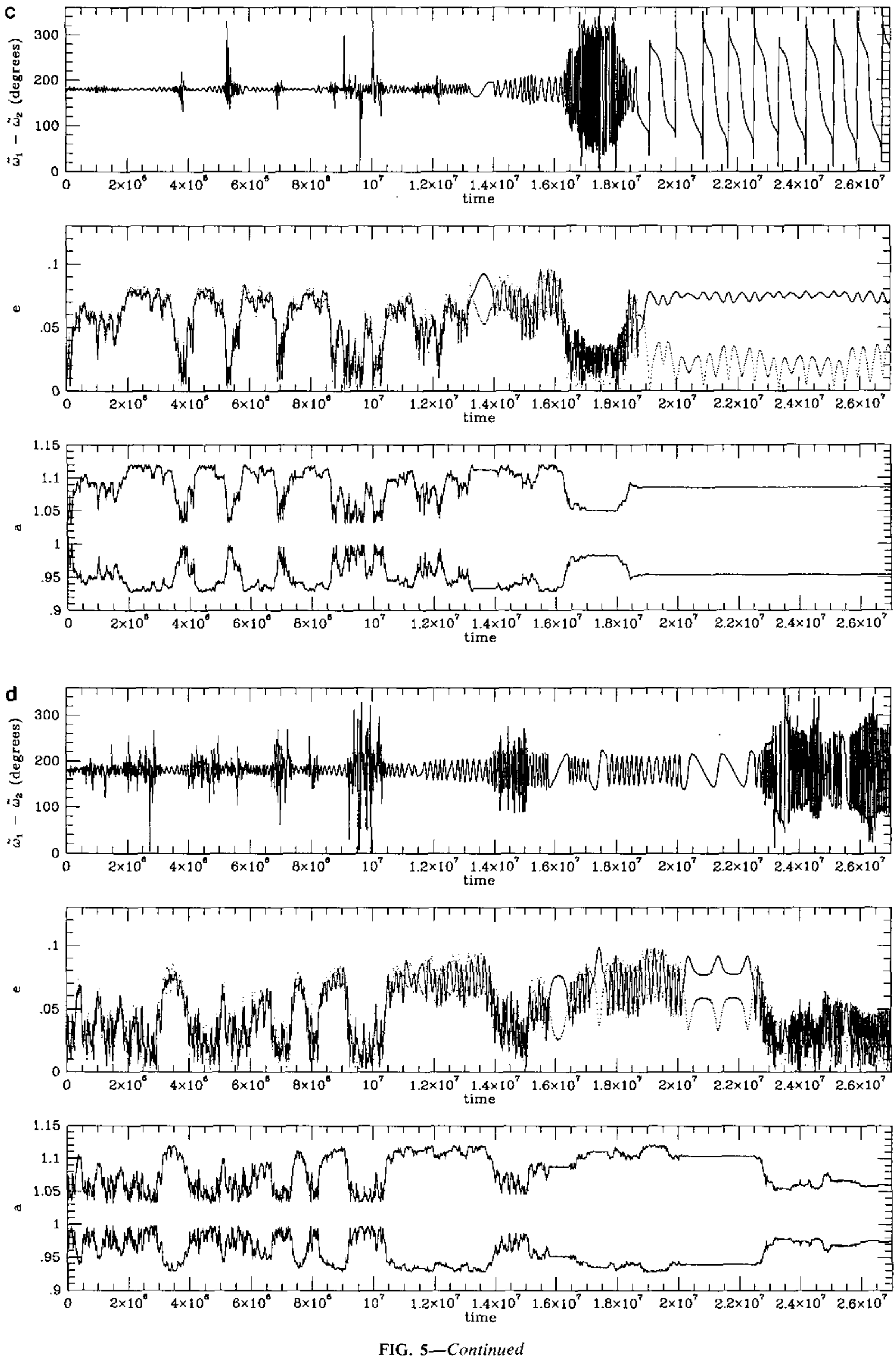

FIG. 5—*Continued*

few seems sufficient) the one-third power of its mass ratio, then the topological criterion is necessary and sufficient for stability. Note that this result is consistent with the circular restricted problem. In that case the smaller planet (the test particle) may have an arbitrarily large initial eccentricity and the Hill curves may still be used to prove stability.

3.4. *Chaotic Dynamics*

The preceding numerical evidence leads us to believe that the topological stability criterion is very valuable for determining whether or not a system on initially circular orbits will actually be stable. That is, if a forbidden surface has not formed the planets will find their way to close encounter. However, what is the detailed behavior of these systems? In order to examine this issue, another set of simulations was conducted with two equal mass planets for the values $\mu = 10^{-5}$, 10^{-6}, 10^{-7}, and 10^{-8} beginning on initially circular orbits. The sizes of the Hill-stable zones were found to agree with Eq. (25) as expected. We shall use the $\mu = 10^{-6}$ case to illustrate the evolution of orbital elements in a two-planet system. For this mass the Hill-stability limit is $\Delta = 0.0306$ (computed using Eq. (23)).

The "crossing zone" of the inner planet shall be defined to be that annulus within which placing the second planet will result in a close approach and thus in general orbit crossing. The outer edge of the crossing zone is given by the Hill-stability limit. Figure 4 shows an evolution inside the crossing zone (with $\Delta = 0.030$). Each abrupt change in orbital elements occurs at a conjunction between the two planets (when the perturbations are largest). Note the size of the sphere of influence is six times smaller than the initial orbital separation. This evolution was terminated when the system came to close approach suddenly at $t \simeq 3100$. The orbital period of the inner planet is $\simeq 2\pi$ in these units, so the system survived about 500 orbits. Also shown is the system's Lyapunov exponent which was calculated by standard methods of phase space shadowing (see Wisdom 1983). Note that the Lyapunov exponent plot indicates that the "Lyapunov time" scale for chaos is, not surprisingly, equal to the synodic period.

Compare this with a system that starts with a slightly larger separation $\Delta = 0.031$, just outside the crossing zone (Fig. 5a). To illustrate the qualitative features, the chaotic nature of the problem, and the wealth of detail present in the dynamics, Figs. 5b–5d show the evolutions for the cases of $\Delta = 0.03101$, 0.03102, and 0.03105, respectively. The Lyapunov time (plot not shown) in all cases is again equal to the synodic period. The evolutions shown have proceeded for more than 200,000 conjunctions of the planets (actually a factor of a few more than this since the semimajor axis separation has increased, shortening the synodic period). We see dramatic excursions in the semimajor axes of the planets (contrary to what secular perturbation theory would predict, as discussed in the introduction). The apparent reflection symmetry in the semimajor axes is nothing more than conservation of energy for the system (see Eq. (16)). One has the impression of a bound on the semimajor axes that is preventing close approaches.

Two other interesting features may be observed in these figures. First notice that the systems occasionally settle into a quiescent state, which can last for millions of orbits. A short integration during this interval might lead to the erroneous conclusion that the system is stable (although the dynamical chaos would be detectable from the Lyapunov exponent). These quiescent periods are always accompanied by smooth eccentricity variations (which are out of phase with each other to conserve angular momentum when the semimajor axes are constant, see Eq. (15)). Such behavior is often seen in heavily interacting systems (see Ipatov 1981, Milani *et al.* 1989).

The second point of interest is that the difference in the perihelion longitudes of the planets indicates that the systems prefer to have their perihelia *anti*-aligned. This might seem unusual because it would seem to allow for a maximum interaction since the perihelion of the outer planet occurs at the same longitude as the aphelion of the inner one. If conjunction occurred at this longitude, the close encounter would be very strong indeed (although the eccentricities are never so large as to have the orbits crossing). Clearly the system has fallen into some sort of resonance protection mechanism that is preventing close approaches at the dangerous longitude. Notice that during quiescent periods the perihelion difference variation can be one of two types: libration about a 180° phase difference or circulation throughout the full range (but with rapid passage through 0° and 180°). In the librating case the passage through $\tilde{\omega}_1 - \tilde{\omega}_2 = 0°$ or 180° is accompanied by a minimum in the eccentricity of one of the planets. This is behavior similar to the $e - \tilde{\omega}$ coupling which functions as a protection mechanism for many asteroids (see Milani and Nobili 1984). Many of the other quiet periods occur near low-order mean-motion commensurabilities which are probably responsible for the maintenance of the resonance (see Greenberg 1977). The details of the protection mechanisms are the subject of ongoing work.

It is hard to imagine a more chaotic planetary system than these examples have shown. This brings us back to the question of the Laplace stability of such systems. Although these systems are Hill stable, nothing prevents the semimajor axis of the outer planet from receding to infinity and it escaping from the system. However, it must do this through a multitude of small encounters since close approaches are forbidden by the Hill stability. Also note that conservation of energy (Eq. (16)) requires that by the time $a_2 \rightarrow \infty$ the semimajor axis of the inner planet has

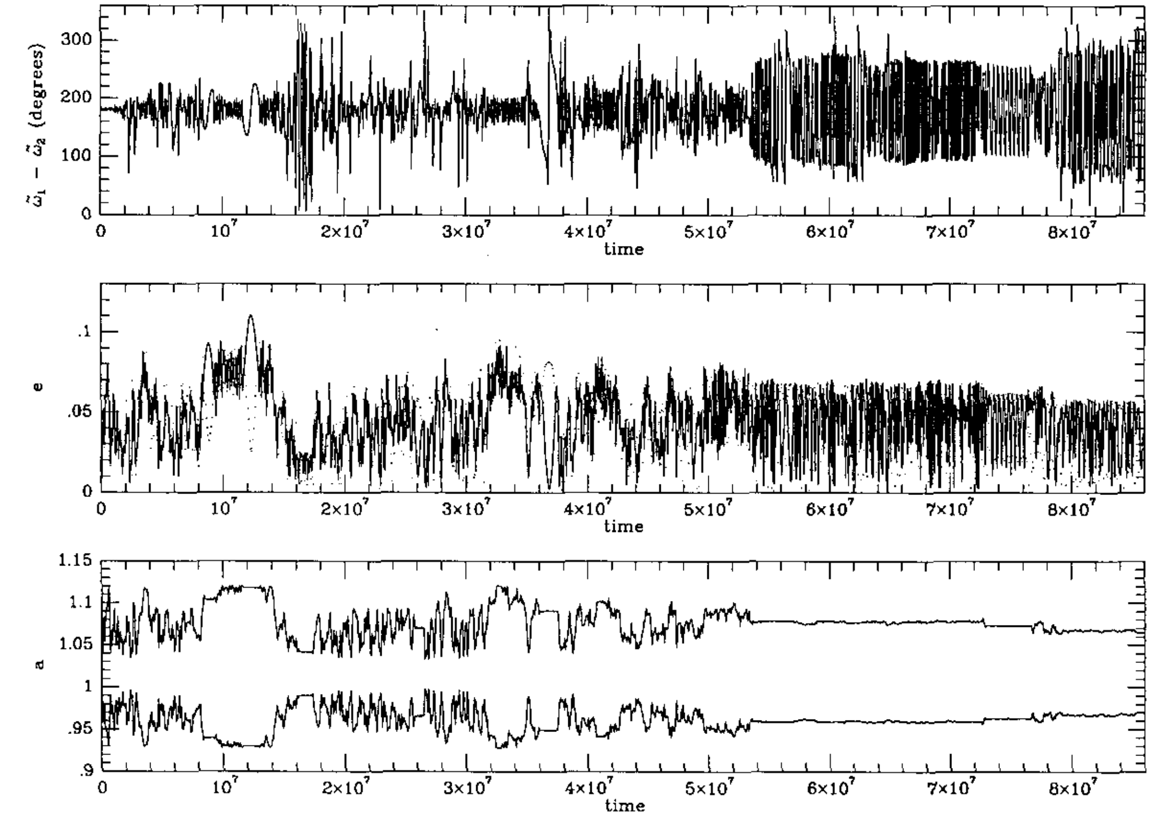

FIG. 6. A longer integration of the system shown in Fig. 5b. There is no evidence of any kind for a trend toward a larger semimajor axis for the outer planet, even over a time scale of millions of conjunctions.

been reduced to

$$a_1 = \frac{1 + \Delta}{1 + \Delta + \mu_2/\mu_1}, \tag{31}$$

which for equal mass planets with a small initial separation requires that the inner planet move to half of its original semimajor axis. Figure 6 shows the longest integration performed (which required a week of dedicated time on a fast IBM workstation). This proceeded for more than 14 million orbits of the inner planet (only a factor of 2 less than the number of orbits Neptune has completed since the origin of the Solar System). There is still no apparent trend toward escape of the outer planet, even though the system has gone through more than a million conjunctions. This perhaps runs counter to the intuition of those who would appeal to the phenomena of Arnold diffusion which demands that the system must eventually visit every chaotic region of the phase space (which includes very high eccentricity orbits for the outer planet). However, we see no apparent trend toward this state in the previous figures, even though the systems have been evolving for more than 10^5 Lyapunov times! Here we have more evidence that Arnold diffusion may have little practical significance in celestial mechanics for problems of this type; the time scale for this phenomena may be simply so long that it is not of practical significance (see Nekhoroshev 1977).

After a certain orbital separation is reached one finds that quasi-periodic orbits appear. That is, there is a zone of "global chaos" closer to the planet. This is analogous again to the behavior of the circular restricted problem, with the size of the chaotic zone varying with the well known scaling of $\mu^{2/7}$ (Wisdom 1980). We can define this zone empirically exactly the same way as the crossing zone was established (i.e., systems with Δ smaller than some critical value Δ_{ch} are all chaotic). Figure 7 shows how an increased separation of the planets eventually results in quasi-periodic orbits. All orbits closer than some critical value ($\Delta_{ch} = 0.035$ in this case) are chaotic.

Figure 8a plots, as a function of the planetary mass ratio, the critical value Δ_{ch} above which global chaos ceases to occur. Plotting the results yields Fig. 8a. The size of the chaotic zone is reasonably well fit by the relation

$$\Delta_{ch} \simeq 2\mu^{2/7}. \tag{32}$$

The coefficient should be obtainable from an analysis

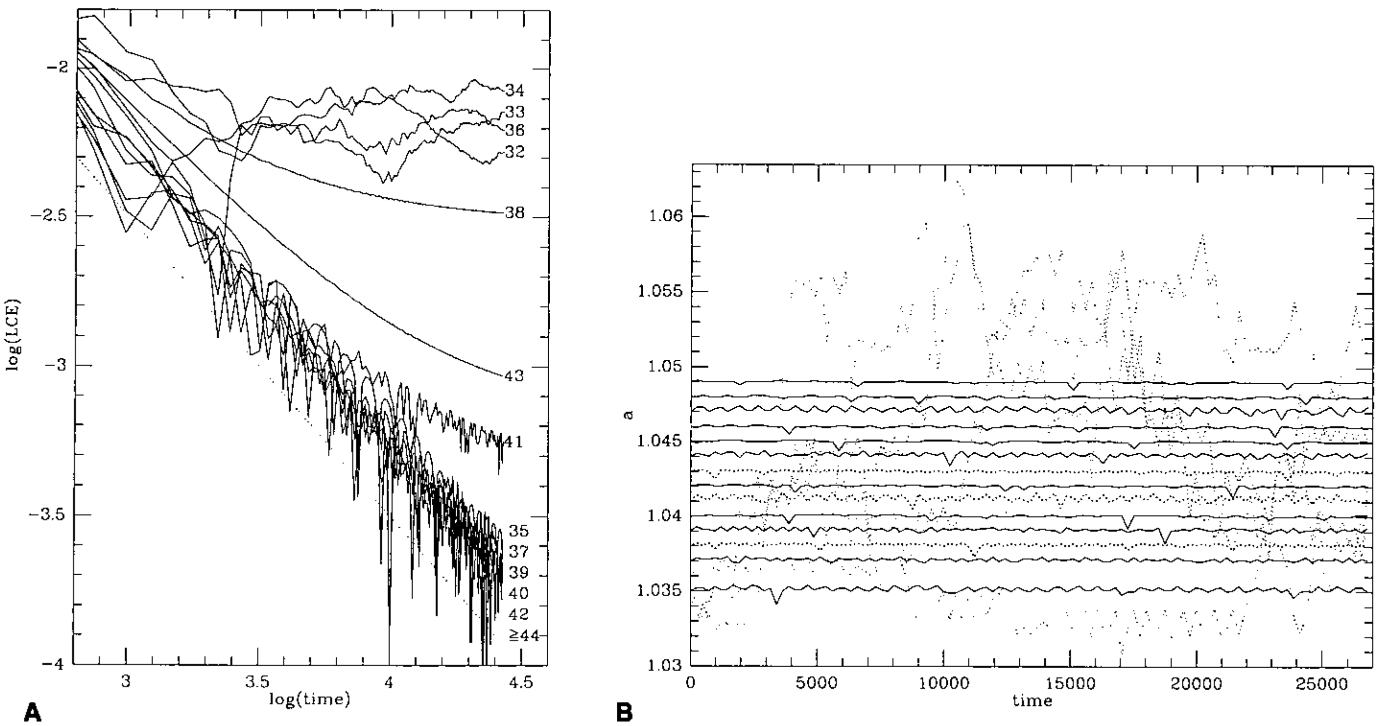

FIG. 7. (a) Lyapunov exponents for a set of simulations in the case $\mu_1 = \mu_2 = 10^{-6}$, $e(t = 0) = 0$. The Lyapunov exponents are labeled with two digit numbers *nn* which denote the initial orbital separation $\Delta = 0.0nn$. The dotted line shows the reference slope of -1 with which quasi-periodic orbits should stay parallel. All orbits with $\Delta < 0.035$ are in the "global chaos" zone. (b) Semimajor axis evolution for the same systems. All chaotic orbits are shown with dotted lines. Observe the three chaotic orbits indicated by heavy dots having semimajor axis changes no larger than the quasi-periodic orbits. One of these is discussed in detail below.

of the resonance overlap criterion. Notice that the chaotic zone is larger than the crossing zone for the mass range of interest. It is perhaps easier to visualize this with a diagram like Fig. 8b. Such a diagram, although an oversimplification, is useful for a first-order understanding of the dynamics of these nearly circular systems.

The outer edge of the crossing zone is very sharply defined by the formation of the Hill forbidden surface. In contrast, the outer edge of the chaotic zone is "fuzzy" since it will be determined by the overlap of resonances. Figure 7 shows that beyond the critical global chaos region there are still some cases of isolated chaotic orbits (presumably due to isolated resonances away from the main overlap region). Also note that the chaotic zone encompasses the crossing zone (planets suffering repeated close encounters will certainly be chaotic) and that there are bound to be some quasi-periodic orbits inside the global chaos zone (a pair of planets on horseshoe orbits furnishes an example of this (cf. Petit and Henon 1986) as well as the occasional stable periodic orbit).

How relevant is the chaos to the stability of these systems? If one examines systems near the outer edge of the chaotic zone, one finds something interesting. Figure 9 compares the orbital element evolution of a system with $\Delta = 0.038$ with one having $\Delta = 0.039$; the first system is chaotic while its neighbor is quasi-periodic. Even though the Lyapunov exponent plots very clearly indicate that the closer ($\Delta = 0.038$) system is *heavily* chaotic and the other is not so, there are no distinguishing characteristics that would lead one to conclude this from the very similar orbital element histories. This is true even though the chaotic system has been evolving for more than 3000 Lyapunov times! There appear to be NO macroscopic effects from the chaos, not even the large changes in semimajor axis observed in the systems closer to the edge of the crossing zone. The system is almost certainly trapped in an isolated chaotic region. What is the mechanism that is producing the chaotic signature? In Fig. 10 a much shorter portion of the evolution of the two systems is presented. In both of these simulations a shadow object was started with a semimajor axis 10^{-8} larger than the outer planet. Although it is below the resolution of the graph, the quasi-periodic system fluctuates around zero element difference with a roughly *constant* amplitude about four orders of magnitude smaller than that shown for the chaotic system. The chaotic system, on the other hand, shows the characteristic exponential divergence from its shadow system. However, note that from the previous figure this exponential divergence does not produce any macroscopic signature in the orbit elements. So

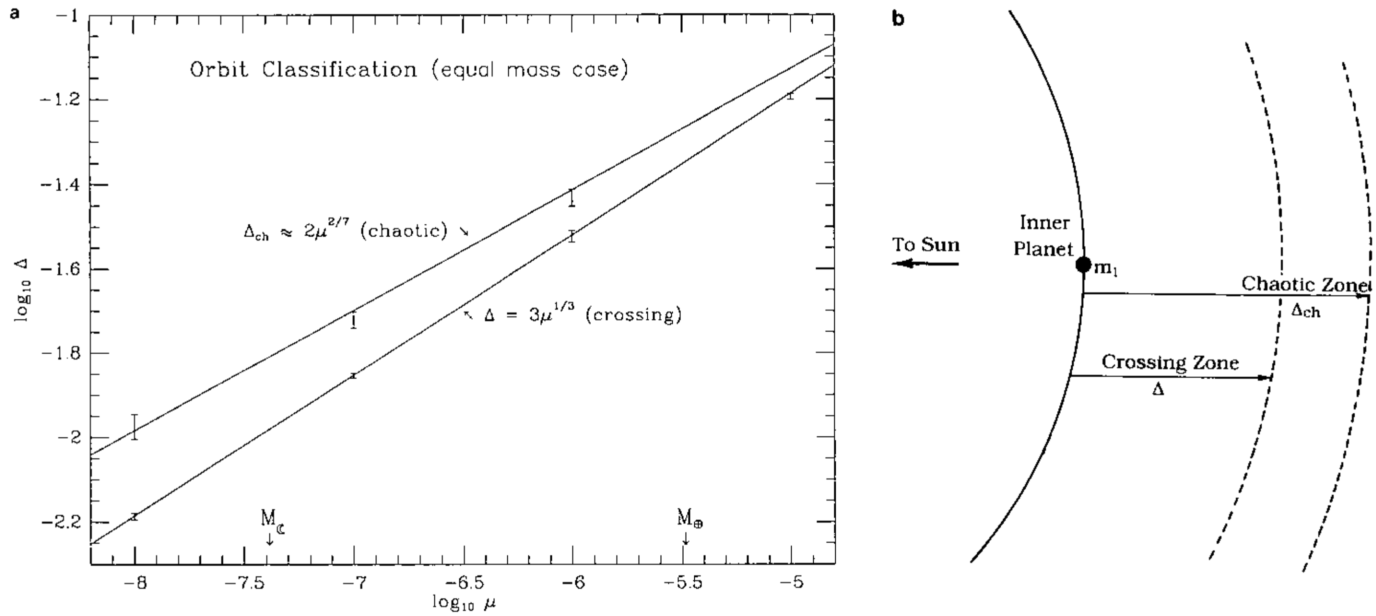

FIG. 8. (a) The size of the chaotic and crossing zones for equal mass planets as a function of mass ratio $\mu = m_1/m_3$. The results of the numerical experiments are shown with an empirical fit for the chaotic zone and the topological stability prediction for the chaoic zone. The mass ratios of the Earth and Moon (relative to the Sun) are shown for comparison. The chaotic zone becomes appreciably larger than the crossing zone only for smaller masses. (b) A more physical picture of the above result. If the outer planet is placed inside the crossing zone annulus it will be Hill unstable, whereas if it is put inside the chaotic zone, it will have a nonzero LCE. See the text for more discussion.

while predicting the exact positions of the planets in the closer system is rendered impossible by the chaos, the stability of the system seems to be unaffected. Thus, *simply observing that the system is chaotic does not necessarily imply crossing or any other large-scale chaotic behavior.* This serves as a counterexample to the results of Lecar *et al.* (1992) which indicate a correlation between the crossing time and the square of the Lyapunov time. The systems examined here can *never* reach crossing orbits; and since there seems to be no macroscopic changes in the orbits, this situation furnishes a quite clear example of "bound chaos" in celestial mechanics problems (Murray 1992).

4. DISCUSSION

Although the model is an oversimplification, the preceding results are pertinent to theories of planetary formation, the current Solar System, and the recently discovered pulsar–planet system.

4.1. Implications for Planetary Formation

According to Eq. (24), the feeding zone of a protoplanet of mass ratio μ_p sweeping up low-eccentricity planetesimals of negligible mass is

$$\Delta_{\text{feed}} = 2.4\mu_p^{1/3} = 2.4 \cdot 3^{1/3}\left(\frac{\mu_p}{3}\right)^{1/3} = 3.5R_{\text{H}}, \quad (33)$$

where R_{H} is the Hill sphere radius. Planetesimals farther away than Δ_{feed} are Hill stable. This explains the previously known empirical results from numerical simulations (Lissauer 1987). Of course, self-stirring by the planetesimal swarm (or other protoplanets) can push the planetesimals into the feeding zone by changing their angular momentum and/or energy. Nor are all planetesimals within this feeding zone necessarily going to be accreted by the planet (for example, those on horseshoe orbits; cf. Greenberg *et al.* 1991).

Two adjacent protoplanets that are formed would likely be outside each others' feeding zones in any case. Assuming that these protoplanets have comparable mass, the previous results show that if their separation is greater than $3\mu^{1/3}$ they cannot coalesce to form a larger planet. This again assumes no other outside influences capable of appreciably changing their c^2h. However, if they are in the chaotic regime, they could still have considerable variation in their semimajor axes (e.g., Fig. 5a begins with an initial separation of about $4.5R_H$ in the semimajor axes of the two planets but has subsequent excursions of $10R_H$. This will have two effects. First, the migrating embryos will be pushed into new, planetesimal-rich zones from which they may accrete more material (although because of their high eccentricities the relative velocities will be large). Second, they may very well be moved so far as to interact heavily with the next neighboring protoplanet. Figure 8a demonstrates that in the inner Solar System, where embryos might have masses of $\mu \sim 10^{-8}$, the cha-

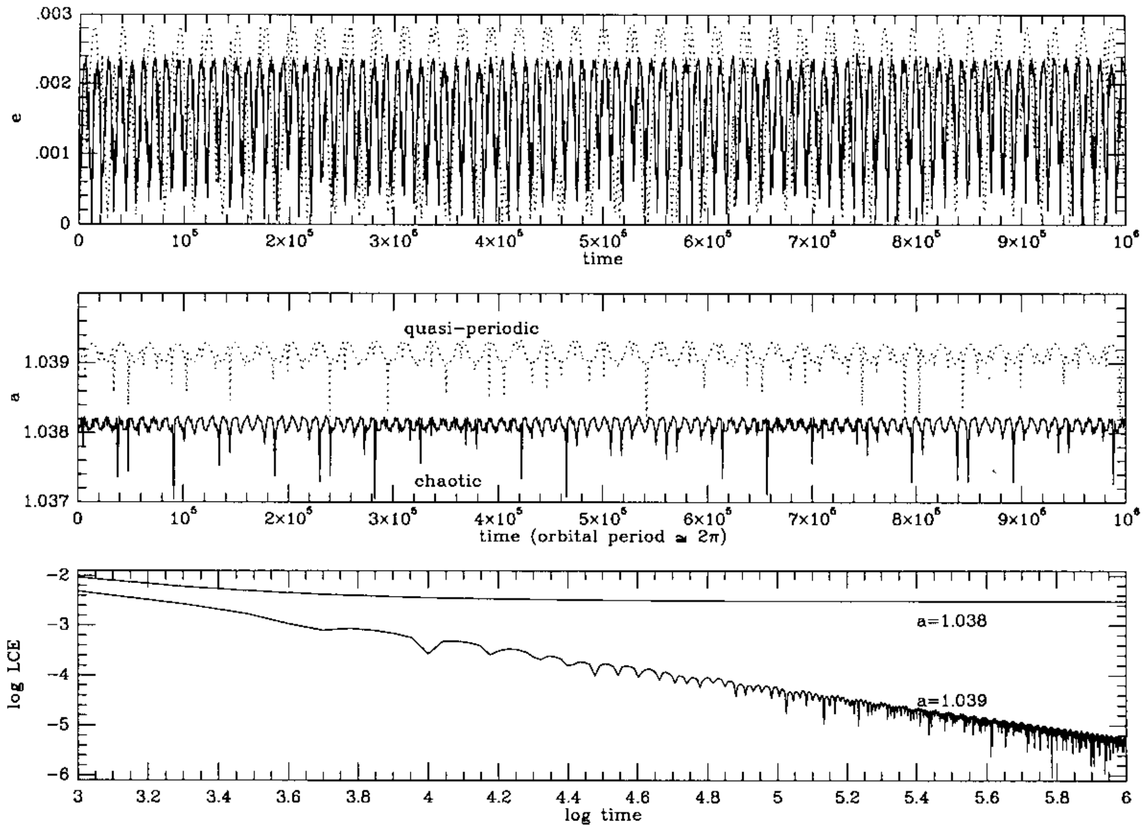

FIG. 9. Comparison of evolutions for the case $\mu_1 = \mu_2 = 10^{-6}$ with $\Delta = 0.038$ and $\Delta = 0.039$. The former is chaotic while the latter is quasi-periodic. The dotted lines show the $\Delta = 0.039$ case.

otic zone can be 50% larger than the crossing zone (about $7R_H$) and the eccentricities produced will quickly put adjacent embryos into crossing orbits. In contrast, in the outer Solar System (if the embryos produced in the runaways are closer to the $\mu \sim 10^{-5}$ range) the chaotic zone is not much larger than the crossing zone. Thus in the outer Solar System the usual Hill separation is all that is required for the temporary cessation of growth of the entire set of embryos. If one views the logical endpoint of the early formation as a whole set of embryos formed through runaway accretion on nearly circular orbits separated by roughly the size of their feeding zones, then one is faced with the following consequence. In the inner Solar System this configuration will rapidly become unstable due to the dynamical chaos (i.e., orbit crossing will take place very quickly) while in the outer Solar System the set of embryos is more stable. Orbit crossing in the outer Solar System will have to wait for secular effects to set it. In either case, once a few embryos are in crossing orbits the strong close encounters will presumably produce high eccentricities that will propagate throughout the swarm ("all hell will break loose").

In summary, a reasonably stable set of planetary embryos would have to be successively separated by more than their chaotic zone sizes. Since this can be quite a bit larger than their feeding zone sizes (by a factor of 2 or more) it is unlikely that the runaway process would produce a set of isolated embryos.

4.2. The Current Solar System

Since the criteria above are for isolated three-body systems, the traditional approach (cf. Milani and Nobili 1983a) when applying these results to the Solar System has been to break the latter into three-body subsystems and to check the Hill stability of these subsystems. One finds that many planetary subsystems can be shown to be stable (for example Sun–Jupiter–Saturn) while others cannot be (Sun–Jupiter–Mercury). Milani and Nobili (1983b) also present some results showing how the presence of a fourth body can produce changes in the c^2h value of a three-body system (and hence destabilize it).

However, one finds that in the case of the Solar System one cannot guarantee the stability of some of the apparently stable subsystems in the Solar System (for example, Sun–Jupiter–Io!). As discussed above, failing the Hill stability test does not imply an empirical instability if either the eccentricities are large and/or the masses of the two smaller bodies are very different. See Milani and

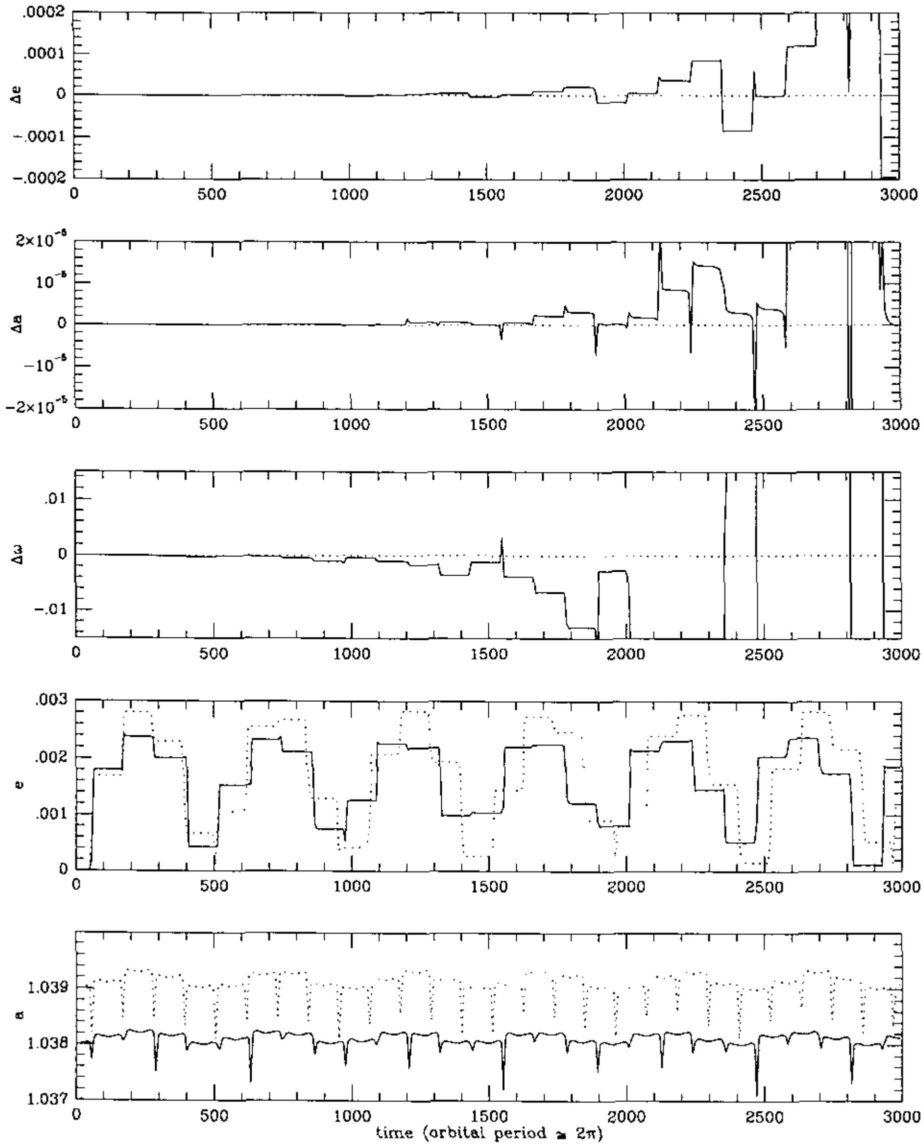

FIG. 10. Divergence of "shadow trajectories" from the phase space trajectory of the outer planet in each of the cases shown in Fig. 9. The top three panels show the difference in orbital elements between the outer planet and its shadow, the dotted line corresponding to the quasi-periodic system (with the larger separation), and the solid line to the chaotic system. Note the exponential divergence of the elements in the chaotic case. See text for discussion.

Nobili (1983a) and Valsecchi *et al.* (1984) for further discussion.

4.3. Mass Constraints on PSR 1257+12

Let us explore what can be said about the stability of the newly discovered pulsar–planet system using the Hill stability criterion. Using the values from Rasio *et al.* (1992) for the system, the current separation of the semi-major axes of the planets is $\Delta = 0.031$, and the mass ratio $\mu_2/\mu_1 = 0.82$. Equation (23) can then be used to compute the maximum mass ratios that would still allow the system to be Hill stable. The result is

$$\mu_1 \simeq 9.0 \times 10^{-4}, \quad \mu_2 \simeq 7.4 \times 10^{-4}, \qquad (34)$$

which are both slightly less than the mass of Jupiter. Note that in this maximum mass case we are justified in neglecting the eccentricities since $e \simeq 0.02 < \mu^{1/3} \simeq 0.1$. The requirement of separation by the size of the chaotic zone is not really relevant for an upper limit on the masses since we have seen above that this will not produce a gross instability of the system (i.e., it will not lead to escape of one of the masses).

This is only an upper limit in the sense that if the masses are below this value then the topological criterion guarantees Hill stability. Masses slightly larger than this value could be empirically stable as discussed earlier. Indeed, numerical experiments (Malhotra 1993, Rasio *et al.* 1992) give an upper limit on the masses of about twice the above values.

Masses as large as this upper limit are unlikely since the system must then be almost pole-on to produce the observed signal. If this were the case, then the mutual perturbations should be very obvious at each conjunction (which occur roughly every 7 months). Therefore, it is likely that the system is Hill stable, with masses down near the minimum values given by the assumption of observation of an edge-on system.

5. CONCLUSION

The topological stability criterion from the three-body problem furnishes a useful stability formula for the case of a system of two planets (although as a test of instability it is less useful for initially eccentric orbits). For initially circular orbits ($e < \mu^{1/3}$) a fractional initial orbital separation of at least

$$\Delta \simeq 2.4(\mu_1 + \mu_2)^{1/3} \tag{35}$$

ensures Hill stability (with the higher-order correction given in Eq. (23)).

Systems with slightly larger orbital separations usually display chaotic behavior. The observed chaotic behavior is confined away from the zone of chaos that contains planet-crossing behavior by the Hill stability surface (which thus behaves as an isolating integral). Many of the chaotic systems show no macroscopic signatures of the chaos in their orbital elements over thousands to millions of Lyapunov times.

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